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The geometry of multi-qubit entanglement

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Abstract

This paper, a continuation of a previous one (Iwai 2007 *J. Phys. A: Math. Theor.* **40** 1361), studies the geometry of multi-qubit entanglement with respect to bipartite partitions. An n-qubit system $(\mathbf{C}^2)^{\otimes n}$ is isomorphic with $\mathbf{C}^{2^\ell} \otimes \mathbf{C}^{2^m} \cong \mathbf{C}^{2^\ell \times 2^m}$, the linear space of $2^\ell \times 2^m$ complex matrices, where $\ell + m = n$. According to the isomorphism, the local transformation group $U(2^\ell) \times U(2^m)$ acts on the space of normalized states in $\mathbf{C}^{2^\ell \times 2^m}$. Let M and G denote the space of normalized states and the local transformation group acting on M, respectively. According to the orbit types, M is stratified into strata, among which a principal stratum and the sets of separable states and maximally entangled states will be identified. For $C \in M \subset \mathbf{C}^{2^\ell \times 2^m}$, a function $F(C) = \det(I - CC^*)$ proves to serve as a measure of entanglement, where I denotes the $2^\ell \times 2^\ell$ identity matrix. The $F(C)$ attains the minimal and the maximal values, respectively, on the sets of separable states and maximally entangled states, and further takes no extremal values on the principal stratum. The $F(C)$ projects to a function on the factor space $G\backslash M$. A naturally defined metric on M also projects to that on $G\backslash M$, which serves to measure the distance between the separable states, $F^{-1}(0)$, and the states, $F^{-1}(k)$, of prescribed value k of measure. To be precise, one has to restrict M to the principal stratum, when projecting the metric. Three- and four-qubit systems will be studied intensively, and then multi-qubit systems discussed.

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1. Introduction

Since concurrence, which is defined to be a measure of entanglement, is closely related with local transformation groups, invariants for the groups has been intensively studied algebraically in particular [1–7]. Geometric study of entanglement has also been studied in various ways [8–12]. This paper, a continuation of a previous paper [13], has an aim to study entanglement from the viewpoint of Riemannian geometry.

An n-qubit system $(\mathbf{C}^2)^{\otimes n}$ is isomorphic with $\mathbf{C}^{2^\ell} \otimes \mathbf{C}^{2^m} \cong \mathbf{C}^{2^\ell \times 2^m}$, the linear space of $2^\ell \times 2^m$ complex matrices with $\ell + m = n$, according to which the local transformation group

$G := U(2^\ell) \times U(2^m)$ acts on the state space $M \cong S^{2^{n+1}-1}$, the space of normalized states in $\mathbf{C}^{2^\ell \times 2^m}$. For a three-qubit system, one has $\ell = 1$ and $m = 2$, so that $2^\ell = 2$ and $2^m = 4$, according to which the local transformation group is given by $U(2) \times U(4)$. For a four-qubit system, there are two partitions such as $(\ell, m) = (1, 3)$ and $(\ell, m) = (2, 2)$, according to which one has local transformation groups $U(2) \times U(8)$ and $U(4) \times U(4)$, respectively.

According to the orbit types of the G action, M is stratified into strata, among which a principal stratum and the sets of separable states and maximally entangled states are identified. For $C \in M$, a function $F(C) = \det(I - CC^*)$ proves to serve as a measure of entanglement. In fact, it attains the minimal and the maximal values, respectively, on the sets of separable states and maximally entangled states, and further takes no extremal values on the principal stratum. Since the function $F(C)$ is invariant under the G action, it projects to a function on the factor space $G\backslash M$. A naturally defined metric on M, which is invariant under the G action as well, projects to a metric on $G\backslash M$, which serves to measure the distance between the separable states, $F^{-1}(0)$, and the states, $F^{-1}(k)$, of prescribed value k of the measure. To be precise, one has to restrict M to the principal stratum $\dot{M}$, when projecting the metric. It will be shown that $G\backslash\dot{M}$ is diffeomorphic with an open submanifold of the sphere $S^{2^{\ell+1}-1}$ and that the metric defined through the submersion $\dot{M} \to G\backslash\dot{M}$ is isometric to the canonical metric on $S^{2^{\ell+1}-1}$. The distance between $F^{-1}(0)$ and $F^{-1}(k)$ is then measured explicitly. Three- and four-qubit systems will be studied intensively. In particular, all types of orbits of G will be identified.

The organization of this paper is as follows: section 2 contains setting up for geometric study of entanglement with respect to a bipartite partition. A measure, $F(C) = \det(I - CC^*)$, of entanglement and the local transformation group $U(2^\ell) \times U(2^m)$ are introduced. The state space M is endowed with a natural Riemannian metric. According to the group action on M, each tangent space to M is decomposed into a direct sum of vertical and horizontal subspaces, where the definition of these subspaces will be given in section 2. The decomposition will be used later for determining the metric on the factor space $G\backslash\dot{M}$. Section 3 deals with three-qubits. The stratification of $M \cong S^{15}$ by the local transformation group $U(2) \times U(4)$ is fully studied. M is stratified into three strata, among which are there the sets of separable states and maximally entangled states. The metric on $G\backslash\dot{M}$ is discussed in association with the concurrence $2\sqrt{\det(CC^*)}$ with $C \in M$. Note here that $\det(CC^*) = \det(I - CC^*)$ in the present case. In sections 4 and 5, four-qubit entanglement is studied according to the isomorphism $(\mathbf{C}^2)^{\otimes 4} \cong \mathbf{C}^4 \otimes \mathbf{C}^4$ and $(\mathbf{C}^2)^{\otimes 4} \cong \mathbf{C}^2 \otimes \mathbf{C}^8$, respectively. Orbit types are fully studied, and entanglement is measured by means of the metric on $G\backslash\dot{M}$ in respective cases. Section 6 contains the study of multi-qubits. The principal stratum $\dot{M}$ and the sets of separable states and maximally entangled states are identified. It will be shown that $G\backslash\dot{M}$ is diffeomorphic with an open submanifold of the sphere $S^{2^{\ell+1}-1}$ and that the metric defined on it is isometric with the canonical metric on $S^{2^{\ell+1}-1}$. The distance is measured between the separable states, $F^{-1}(0)$, and the states, $F^{-1}(k)$, of prescribed value k of the measure. Section 7 contains concluding remarks.

2. Setting up

We start with an n-qubit state of the form

$$\Psi = \sum_{j_1,\ldots,j_n \in \{0,1\}} c_{j_1\cdots j_n} \boldsymbol{e}_{j_1} \otimes \boldsymbol{e}_{j_2} \otimes \cdots \otimes \boldsymbol{e}_{j_n}, \tag{2.1}$$

where $\sum_{j_1,\ldots,j_n} |c_{j_1\cdots j_n}|^2 = 1$, and $\boldsymbol{e}_j$ are basis vectors of the standard basis of $\mathbf{C}^2$. If the n-qubit state is separable, it is decomposed into a tensor product of ℓ-qubit and the other

m-qubit with $\ell + m = n$. With this in mind, we rewrite Ψ as

$$\Psi = \sum_{A,K} c_{AK} \boldsymbol{e}_A \otimes \boldsymbol{e}_K, \tag{2.2}$$

where the index $A = j_1 \cdots j_\ell$ and $K = j_{\ell+1} \cdots j_{\ell+m}$ are binary integers, and

$$\boldsymbol{e}_A = \boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_\ell}, \qquad \boldsymbol{e}_K = \boldsymbol{e}_{j_{\ell+1}} \otimes \cdots \boldsymbol{e}_{j_{\ell+m}}. \tag{2.3}$$

Then, the map $\left(c_{j_1\cdots j_n}\right) \mapsto C = (c_{AK})$ provides the isomorphism

$$(\mathbf{C}^2)^{\otimes n} \cong \mathbf{C}^{2^\ell \times 2^m}, \tag{2.4}$$

where $\mathbf{C}^{2^\ell \times 2^m}$ denotes the linear space of $2^\ell \times 2^m$ complex matrices. Since the state Ψ is normalized, C is subject to the constraint $\mathrm{tr}(CC^*) = 1$. Thus, the state space for an n-qubit system is expressed as

$$M = \{C \in \mathbf{C}^{2^\ell \times 2^m} | \mathrm{tr}(CC^*) = 1\}, \tag{2.5}$$

which is diffeomorphic with the sphere $S^{2^{n+1}-1}$. M is endowed with a Riemannian metric through

$$\langle X_1, X_2 \rangle = \tfrac{1}{2}\, \mathrm{tr}(X_1^* X_2 + X_2^* X_1), \qquad X_1, X_2 \in T_C M, \tag{2.6}$$

where $T_C M$ denotes the tangent space to M at C,

$$T_C M = \{X \in \mathbf{C}^{2^\ell \times 2^m} \mid \mathrm{tr}(C^* X + X^* C) = 0\}. \tag{2.7}$$

Now, the state Ψ is separable in the sense that Ψ is a tensor product of the first ℓ-qubit and the last m-qubit, if and only if C is of rank 1. On account of the constraint $\mathrm{tr}(CC^*) = 1$ and the fact that CC^* is positive semi-definite, the matrix CC^* is of rank 1, if and only if $\det(I - CC^*) = 0$, where I denotes the $2^\ell \times 2^\ell$ identity matrix. Since C and CC^* have the same rank, we may take the function

$$F(C) = \det(I - CC^*) \tag{2.8}$$

as a measure of entanglement. By definition, one has $F(C) = 0$ for the separable states C. Let $\lambda_1, \ldots, \lambda_{2^\ell}$ be the eigenvalues of CC^*. Then, one has $F(C) = (1 - \lambda_1) \cdots (1 - \lambda_{2^\ell})$, which takes the maximal value when all of λ_j are equal to one another. In view of this and the fact that CC^* is associated with the partial trace of a density matrix $|\Psi\rangle\langle\Psi|$ with respect to last m-qubit, the states on which $F(C)$ takes the maximal value may be called maximally entangled. However, it is to be noted that the bipartite partition is not unique, so that the entanglement measure $F(C)$ depends on bipartite partitions. We note further that the function $F(C)$ is a natural extension of the concurrence. In fact, for three-qubits with $(\ell, m) = (1, 2)$, one has $\det(I - CC^*) = \det(CC^*)$, as is easily verified, and the concurrence for three-qubits is defined to be $2\sqrt{\det(CC^*)}$, according to [14].

The group $G := U(2^\ell) \times U(2^m)$ acting on $\mathbf{C}^{2^\ell} \otimes \mathbf{C}^{2^m}$ determines an action of G on $M \subset \mathbf{C}^{2^\ell \times 2^m}$, which is described as

$$C \mapsto gCh^T, \qquad (g, h) \in U(2^\ell) \times U(2^m), \tag{2.9}$$

where the superscript T denotes the transposition. According to the orbit types of the G action, M is stratified into strata, among which the sets of separable states and maximally entangled states are identified. Further, it is easy to see that $F(C) = \det(I - CC^*)$ is invariant under the G action. Hence, $F(C)$ will project to a function on the factor space $G\backslash M$. However, we have to point out that $G\backslash M$ is not a manifold in general. From the viewpoint of Riemannian geometry for entanglement measurement, we are interested in the principal stratum $\dot{M}$ and in the Riemannian metric on $\dot{M}$, which is also invariant under the G action, and then projects to a metric on $G\backslash\dot{M}$.

According to the action of $G = U(2^\ell) \times U(2^m)$ on M, the tangent space $T_C M$ is decomposed into the sum of vertical and horizontal subspaces, where the vertical subspace V_C and the horizontal subspace H_C of $T_C M$ are defined to be the tangent space to the orbit $\mathcal{O}_C$ through C and to be the orthogonal complement to V_C with respect to the Riemannian metric on M, respectively. From the definition, V_C and H_C are shown to be expressed as

$$V_C = \{\xi C + C\eta^T \mid \xi \in u(2^\ell), \eta \in u(2^m)\} \cong \mathcal{G}/\mathcal{G}_C, \tag{2.10}$$

$$H_C = \{X \in T_C M \mid CX^* - XC^* = 0, C^*X - X^*C = 0\}, \tag{2.11}$$

where $\mathcal{G}$ and $\mathcal{G}_C$ denote the Lie algebras of the group G and the isotropy subgroup G_C, respectively. In particular, we will take up H_C for $C \in \dot{M}$ in section 6 to determine a Riemannian metric on $G\backslash\dot{M}$ through the projection $\dot{M} \to G\backslash\dot{M}$. The entanglement will be measured by means of this metric together with the function F projected onto $G\backslash\dot{M}$.

3. Three-qubit systems

3.1. Isotropy subgroups

We work with a three-qubit system with $(\ell, m) = (1, 2)$. To study the isotropy subgroup of $G = U(2) \times U(4)$ acting on the state space $M \subset \mathbf{C}^{2\times 4}$, we decompose $C \in \mathbf{C}^{2\times 4}$ singularly into

$$C = UC_0V^*, \qquad (U, V) \in U(2) \times U(4), \tag{3.1}$$

where

$$C_0 = (\Lambda, 0), \qquad \Lambda = \mathrm{diag}(\mu_1, \mu_2), \quad \mu_1 \geqslant \mu_2 \geqslant 0, \tag{3.2}$$

and μ_j are the singular values of C. If $gC_0h^T = C_0$ for $(g, h) \in U(2) \times U(4)$, then one has $UgU^{-1}C(\overline{V}hV^T)^T = C$. This implies that the isotropy subgroups G_{C_0} at C_0 and G_C at C are isomorphic to each other. Hence, our task reduces to finding G_{C_0}. Suppose that $gC_0h^T = C_0$. We put h in the form, $h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}$, where $h_1, \ldots, h_4 \in \mathbf{C}^{2\times 2}$. Then, we obtain $g\Lambda h_1^T = \Lambda$ and $g\Lambda h_3^T = 0$. If $\det\Lambda \neq 0$, then $h_3 = 0$. From the condition $hh^* = I_4$, one obtains $h_2 = 0$, and $h_1h_1^* = h_4h_4^* = I_2$. According to whether $\mu_1 \neq \mu_2$ or $\mu_1 = \mu_2$, the condition $g\Lambda h_1^T = \Lambda$ with $\det\Lambda \neq 0$ implies that

$$g = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}), \quad h_1 = g^{-1} \quad \text{or} \quad g \in U(2), \quad h_1 = (g^{-1})^T. \tag{3.3}$$

Hence, under the condition $\det\Lambda \neq 0$, according to whether $\mu_1 \neq \mu_2$ or $\mu_1 = \mu_2$, one has

$$g = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_1} & \\ & \mathrm{e}^{\mathrm{i}\theta_2} \end{pmatrix}, \qquad h = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta_1} & & \\ & \mathrm{e}^{-\mathrm{i}\theta_2} & \\ & & h_4 \end{pmatrix}, \quad h_4 \in U(2), \tag{3.4}$$

or

$$g \in U(2), \qquad h = \begin{pmatrix} (g^{-1})^T & \\ & h_4 \end{pmatrix}, \qquad h_4 \in U(2). \tag{3.5}$$

Let $\det\Lambda = 0$. Then, one has $\mu_1 = 1$ and $\mu_2 = 0$. We put h in the form $h = \begin{pmatrix} h_0 & h_1 \\ h_2 & h_3 \end{pmatrix}$, where $h_0 \in \mathbf{C}$, $h_1 \in \mathbf{C}^{1\times 3}$, $h_2 \in \mathbf{C}^{3\times 1}$ and $h_3 \in \mathbf{C}^{3\times 3}$. Then, the condition $gC_0h^T = C_0$ yields

$$g = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_1} & \\ & \mathrm{e}^{\mathrm{i}\theta_2} \end{pmatrix}, \qquad h = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta_1} & \\ & h_3 \end{pmatrix}, \qquad h_3 \in U(3). \tag{3.6}$$

Thus, we obtain

$$G_C \cong \begin{cases} U(1)\times U(1)\times U(3), & \text{if} \quad \det(CC^*) = 0, \\ U(1)\times U(1)\times U(2), & \text{if} \quad \det(CC^*) \neq 0, \quad \mu_1(C) > \mu_2(C), \\ U(2)\times U(2), & \text{if} \quad \det(CC^*) \neq 0, \quad \mu_1(C) = \mu_2(C), \end{cases} \tag{3.7}$$

where $\mu_j(C)$ denote the singular values of C.

3.2. The tangent space decomposition

We write the vertical and horizontal subspaces, V_C and H_C, given in (2.10) and (2.11) with $(\ell, m) = (1, 2)$. We first look into V_C. Since $V_C \cong V_{C_0} \cong \mathcal{G}/\mathcal{G}_{C_0}$, we deal with $\mathcal{G}_{C_0}$ for C_0 given in (3.2). If $\det\Lambda = 0$, equation (3.6) implies that $(\xi, \eta) \in \mathcal{G}_{C_0}$ is put in the form

$$\xi = \begin{pmatrix} \mathrm{i}\theta_1 & \\ & \mathrm{i}\theta_2 \end{pmatrix}, \qquad \eta = \begin{pmatrix} -\mathrm{i}\theta_1 & \\ & \zeta_3 \end{pmatrix}, \qquad \zeta_3 \in u(3). \tag{3.8}$$

If $\det\Lambda = \mu_1\mu_2 \neq 0$ and if $\mu_1 \neq \mu_2$, $(\xi, \eta) \in \mathcal{G}_{C_0}$ proves, from (3.4), to be of the form

$$\xi = \begin{pmatrix} \mathrm{i}\theta_1 & \\ & \mathrm{i}\theta_2 \end{pmatrix}, \qquad \eta = \begin{pmatrix} -\mathrm{i}\theta_1 & & \\ & -\mathrm{i}\theta_2 & \\ & & \zeta_2 \end{pmatrix}, \qquad \zeta_2 \in u(2). \tag{3.9}$$

If $\det\Lambda \neq 0$ and if $\mu_1 = \mu_2$, $(\xi, \eta) \in \mathcal{G}_{C_0}$ becomes, from (3.5), of the form

$$\xi \in u(2), \qquad \eta = \begin{pmatrix} -\xi^T & \\ & \zeta_2 \end{pmatrix}, \qquad \zeta_2 \in u(2). \tag{3.10}$$

$\mathcal{G}_{C_0}$ gives rise to vanishing tangent vectors, $\xi C_0 + C_0\eta^T = 0$, in respective cases. $V_{C_0} \cong \mathcal{G}/\mathcal{G}_{C_0}$ is now easy to obtain, but we do not need to give them explicitly.

We now seek for a basis of the horizontal subspace H_{C_0}. Let $X \in H_{C_0}$ be of the form $X = (A, B)$ with $A, B \in \mathbf{C}^{2\times 2}$. Then, from $C_0^* X - X^* C_0 = 0$, one has $A^*\Lambda = \Lambda A$ and $\Lambda B = 0$. If $\det\Lambda \neq 0$, one obtains $B = 0$. From $C_0 X^* - X C_0^* = 0$, one obtains a similar equation, $A\Lambda = \Lambda A^*$. A straightforward calculation then shows that among

$$A_1 = \begin{pmatrix} \mu_2 & 0 \\ 0 & -\mu_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & -\mathrm{i}\mu_2 \\ \mathrm{i}\mu_1 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & \mu_2 \\ \mu_1 & 0 \end{pmatrix}, \tag{3.11}$$

A_1 or A_1, A_2, A_3 are solutions, depending on whether $\mu_1 \neq \mu_2$ or $\mu_1 = \mu_2$. Hence, $X_k = (A_k, 0) \in H_{C_0}$, if $X_k \in T_{C_0}M$. However, we can easily verify that $\mathrm{tr}(C_0^* X_k + X_k^* C_0) = 0$, a condition for $X_k \in T_{C_0}M$, $k = 1, 2, 3$. Thus, we have found that

$$\begin{aligned} &X_1 = (A_1, 0) \in H_{C_0}, && \text{if} \quad \mu_1 > \mu_2, \\ &X_k = (A_k, 0) \in H_{C_0}, k = 1, 2, 3, && \text{if} \quad \mu_1 = \mu_2. \end{aligned} \tag{3.12}$$

In the case of $\det\Lambda = 0$, one has $\Lambda = \mathrm{diag}(1, 0)$, so that $C_0 = (\boldsymbol{e}_1, 0) \in \mathbf{C}^{2\times 4}$ with $\boldsymbol{e}_1 = (1, 0)^T$. We put $Y \in T_{C_0}(M)$ in the form $Y = (\boldsymbol{a}, B)$ with $\boldsymbol{a} \in \mathbf{C}^2$, $B \in \mathbf{C}^{2\times 3}$. Then, the equations $C_0^* Y - Y^* C_0 = 0$ and $C_0 Y^* - Y C_0^* = 0$ together with $\mathrm{tr}(C_0^* Y + Y^* C_0) = 0$ are solved by

$$\boldsymbol{a} = 0, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ b_1 & b_2 & b_3 \end{pmatrix}, \qquad b_k \in \mathbf{C}. \tag{3.13}$$

Thus we have found a basis of H_{C_0},

$$\begin{aligned} &Y_1 = (0, \boldsymbol{e}_2, 0, 0), && Y_2 = (0, 0, \boldsymbol{e}_2, 0), && Y_3 = (0, 0, 0, \boldsymbol{e}_2), \\ &Y_4 = (0, \mathrm{i}\boldsymbol{e}_2, 0, 0), && Y_5 = (0, 0, \mathrm{i}\boldsymbol{e}_2, 0), && Y_6 = (0, 0, 0, \mathrm{i}\boldsymbol{e}_2). \end{aligned} \tag{3.14}$$

Note that if $\mu_1 = 1, \mu_2 = 0$, then $X_1 = -Y_1$. It then turns out that

$$H_{C_0} = \begin{cases} \mathrm{span}\{X_1\}, & \text{if} \quad \det\Lambda \neq 0, \quad \mu_1 \neq \mu_2, \\ \mathrm{span}\{X_1, X_2, X_3\}, & \text{if} \quad \det\Lambda \neq 0, \quad \mu_1 = \mu_2, \\ \mathrm{span}\{Y_1, \ldots, Y_6\}, & \text{if} \quad \det\Lambda = 0, \quad \text{or if} \quad \mu_1 = 1, \mu_2 = 0. \end{cases} \tag{3.15}$$

3.3. The stratification of the state space

According to (3.7), M is stratified into

$$M = M_0 \cup M_1 \cup M_2, \tag{3.16}$$

where

$$\begin{aligned} M_0 &= \{C \in M \mid \det(CC^*) = 0\}, \\ M_1 &= \{C \in M \mid \det(CC^*) \neq 0, \quad \mu_1(C) > \mu_2(C)\}, \\ M_2 &= \{C \in M \mid \det(CC^*) \neq 0, \quad \mu_1(C) = \mu_2(C)\}. \end{aligned} \tag{3.17}$$

M_0 and M_2 are the sets of separable states and maximally entangled states, respectively, with respect to the bipartite partition $(\mathbf{C}^2)^{\otimes 3} \cong \mathbf{C}^2 \otimes \mathbf{C}^4$. M_1 is the principal stratum.

We now study the topology of the strata $M_k, k = 0, 1, 2$. Let $C \in M_0$. Then, C is decomposed into $C = UC_0V^*$, where $(U, V) \in U(2) \times U(4)$, and $C_0 = (\Lambda_0, 0)$ with $\Lambda_0 = \mathrm{diag}(1, 0)$. We put matrices U and V in the form $U = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ and $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_4)$, respectively, where $\boldsymbol{u}_j \in \mathbf{C}^2$ and $\boldsymbol{v}_k \in \mathbf{C}^4$. In this notation, C is expressed as

$$C = \boldsymbol{u}_1\boldsymbol{v}_1^*. \tag{3.18}$$

Since $\boldsymbol{u}_1\boldsymbol{v}_1^* = \boldsymbol{u}_1\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta}\boldsymbol{v}_1^*$, and since $\boldsymbol{u}_1 \in S^3 \subset \mathbf{C}^2$, $\boldsymbol{v}_1 \in S^7 \subset \mathbf{C}^4$, an equivalence relation is defined on $S^3 \times S^7$ by

$$(\boldsymbol{u}_1, \boldsymbol{v}_1) \sim (\boldsymbol{u}_1\,\mathrm{e}^{\mathrm{i}\theta}, \boldsymbol{v}_1\,\mathrm{e}^{\mathrm{i}\theta}). \tag{3.19}$$

Hence, we find that

$$M_0 \cong S^3 \times_{U(1)} S^7, \tag{3.20}$$

which is viewed as a fiber bundle over $S^2 \cong S^3/SO(2)$ with fiber S^7. Put another way, the set of separable states with respect to the bipartite partition forms a submanifold diffeomorphic with $S^3 \times_{U(1)} S^7$.

We turn to M_2. For $C \in M_2$, one has $C = UC_0V^*$, where $(U, V) \in U(2) \times U(4)$, and $C_0 = (\Lambda_2, 0)$ with $\Lambda_2 = \frac{1}{\sqrt{2}}\mathrm{diag}(1, 1)$. Then C is put in the form

$$C = \frac{1}{\sqrt{2}}UV_2^*, \qquad V_2 = (\boldsymbol{v}_1, \boldsymbol{v}_2). \tag{3.21}$$

Since $UV_2^* = UWW^*V_2^*$ for $W \in U(2)$, an equivalence relation is defined on $U(2) \times V_2(\mathbf{C}^4)$ through

$$(U, V_2) \sim (UW, V_2W), \tag{3.22}$$

where $V_2(\mathbf{C}^4) \cong U(4)/U(2)$ denotes the Stiefel manifold of orthonormal two frames in $\mathbf{C}^4$. Hence, we verify that

$$M_2 \cong U(2) \times_{U(2)} V_2(\mathbf{C}^4) \cong V_2(\mathbf{C}^4). \tag{3.23}$$

This means that the set of maximally entangled states forms a submanifold diffeomorphic with $V_2(\mathbf{C}^4)$.

We take up $C \in M_1$. Since $C = UC_0V^*$, where $C_0 = (\Lambda_1, 0)$ and $\Lambda_1 = \mathrm{diag}(\mu_1, \mu_2)$, C is written as

$$C = U(\Lambda_1, 0)V^* = \mu_1 \boldsymbol{u}_1 \boldsymbol{v}_1^* + \mu_2 \boldsymbol{u}_2 \boldsymbol{v}_2^*, \qquad \mu_1 > \mu_2. \tag{3.24}$$

Since $\mu_1 \boldsymbol{u}_1 \boldsymbol{v}_1^* + \mu_2 \boldsymbol{u}_2 \boldsymbol{v}_2^* = \mu_1 \boldsymbol{u}_1 \,\mathrm{e}^{\mathrm{i}\theta_1}\,\mathrm{e}^{-\mathrm{i}\theta_1} \boldsymbol{v}_1^* + \mu_2 \boldsymbol{u}_2 \,\mathrm{e}^{\mathrm{i}\theta_2}\,\mathrm{e}^{-\mathrm{i}\theta_2} \boldsymbol{v}_2^*$, the equivalence relation

$$(U, V_2) \sim (Uh, V_2h), \qquad h = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}) \in U(1) \times U(1) \tag{3.25}$$

is defined on $U(2) \times V_2(\mathbf{C}^4)$, where $V_2 = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ are the same as in (3.21). Hence, we verify that the orbit through $C \in M_1$ is expressed as

$$\mathcal{O}_C \cong U(2) \times_{U(1)\times U(1)} V_2(\mathbf{C}^4), \tag{3.26}$$

which is viewed as a fiber bundle over $S^2 \cong U(2)/(U(1) \times U(1))$ with fiber $V_2(\mathbf{C}^4)$.

To get an idea of the topology of M_1, we consider functions

$$F(C) = \det(CC^*), \qquad f(C) = 2\sqrt{\det(CC^*)}, \tag{3.27}$$

which are invariant under the action of $U(2) \times U(4)$, as is easily verified. Note also that $\det(I - CC^*) = \det(CC^*)$ on account of $\mathrm{tr}(CC^*) = 1$ in this case, and that $f(C)$ is equal to the concurrence defined by Coffman, Kundu and Wootters [14]. Now, for μ_k^2, the squared singular values, one has

$$\det(CC^*) = \mu_1^2\mu_2^2 \leqslant \left(\frac{\mu_1^2 + \mu_2^2}{2}\right)^2 = \left(\frac{1}{2}\,\mathrm{tr}(CC^*)\right)^2 = \frac{1}{4}, \tag{3.28}$$

where the equality occurs if and only if $\mu_1 = \mu_2$. This implies that the range of F is $0 \leqslant F(C) \leqslant 1/4$, so that $0 \leqslant f(C) \leqslant 1$. Thus we have found that the concurrence is a surjective map onto the closed interval $[0, 1]$:

$$f : M \longrightarrow [0, 1]. \tag{3.29}$$

In particular, one has

$$f(M_0) = \{0\}, \qquad f(M_2) = \{1\}. \tag{3.30}$$

Moreover, for $C \in M_1$, we have $0 < f(C) < 1$.

We now assume that $f(C_1) = f(C_2)$ or $F(C_1) = F(C_2)$ for $C_1, C_2 \in M_1$. Since $\mu_1(C_1) > \mu_2(C_1)$ and $\mu_1(C_2) > \mu_2(C_2)$, and since $\mathrm{tr}(C_1C_1^*) = \mathrm{tr}(C_2C_2^*) = 1$, one obtains $\mu_k(C_1) = \mu_k(C_2)$, $k = 1, 2$. Hence, C_1 and C_2 are expressed as $C_1 = g_1(\Lambda_1, 0)h_1^*$ and $C_2 = g_2(\Lambda_1, 0)h_2^*$, respectively, where $(g_1, h_1), (g_2, h_2) \in U(2) \times U(4)$ and $\Lambda_1 = \mathrm{diag}(\mu_1, \mu_2)$ with $1 > \mu_1 > \mu_2 > 0$. From this, it follows that $C_2 = g_2g_1^{-1}C_1\big(h_2h_1^{-1}\big)^*$, which implies that C_1 and C_2 lie in the same orbit of $G = U(2) \times U(4)$. It then turns out that

$$(0, 1) \cong G\backslash M_1, \qquad G = U(2) \times U(4), \tag{3.31}$$

where $(0, 1)$ denotes an open interval. Hence,

$$M_1 \cong (0, 1) \times U(2) \times_{U(1)\times U(1)} V_2(\mathbf{C}^4). \tag{3.32}$$

Equations (3.30) and (3.31) are put together to yield the homeomorphism

$$G\backslash M \cong [0, 1]. \tag{3.33}$$

3.4. Concurrence as a measure of entanglement

We find a metric on $(0, 1) \cong G\backslash M_1$, which is defined through the projection $f : M_1 \to (0, 1)$ from that on $M_1 \subset M$. For notational simplicity, we first treat the function $F(C) = \det(CC^*)$. Since $F(C)$ is invariant under the $G = U(2) \times U(4)$ action, the gradient ∇F should be horizontal; $(\nabla F)_C \in H_C$. We verify this fact in what follows. The gradient ∇F is defined through

$$\mathrm{d}F = \langle \nabla F, \mathrm{d}C \rangle = \tfrac{1}{2}\,\mathrm{tr}((\nabla F)^* \mathrm{d}C + \mathrm{d}C^* \nabla F), \tag{3.34}$$

where $\mathrm{d}C$ should be subject to the constraint $\mathrm{tr}(\mathrm{d}CC^* + C\mathrm{d}C^*) = 0$. If $\det(CC^*) \neq 0$ or if $C \in M_1$, we obtain, leaving the constraint out of account,

$$\mathrm{d}F = \mathrm{tr}(F(C)(CC^*)^{-1}C\mathrm{d}C^* + \mathrm{d}C\,F(C)C^*(CC^*)^{-1}). \tag{3.35}$$

Taking into account the above equations along with the constraint $\mathrm{tr}(\mathrm{d}CC^* + C\mathrm{d}C^*) = 0$, we are allowed to put ∇F in the form $\nabla F = 2F(C)(CC^*)^{-1}C + 2\lambda C$, where λ is a Lagrange multiplier. λ is determined by the condition that ∇F be tangent to M. A calculation results in $\lambda = -2F(C)$. Thus, we have found that

$$\nabla F = 2F(C)((CC^*)^{-1}C - 2C), \qquad C \in M_1. \tag{3.36}$$

It is now easy to verify that ∇F satisfies the conditions given in (2.11).

We proceed to study a metric to be defined on $G\backslash M_1 \cong (0, 1)$. To begin with, we calculate $(\mathrm{d}F)_{C_0}(X_1)$ for $X_1 \in H_{C_0}$, where X_1 is given in (3.12) and $C_0 = (\Lambda_1, 0)$ with $\Lambda_1 = \mathrm{diag}(\mu_1, \mu_2)$. A straightforward calculation provides

$$(\mathrm{d}F)_{C_0}(X_1) = \big\langle (\nabla F)_{C_0}, X_1 \big\rangle_{C_0} = f(C_0)\sqrt{1 - f(C_0)^2}. \tag{3.37}$$

For $(\mathrm{d}F)_C(\widetilde{X}_1)$ with $\widetilde{X}_1 = UX_1V^*$ and $C = UC_0V^*$, we have $(\mathrm{d}F)_C(\widetilde{X}_1) = (\mathrm{d}F)_{C_0}(X_1)$, so that

$$(\mathrm{d}F)_C(\widetilde{X}_1) = r\sqrt{1 - r^2}, \tag{3.38}$$

where we have used the fact that $f(C) = f(C_0) = r$. Since $f = 2\sqrt{F}$, one has $\mathrm{d}f = \mathrm{d}F/\sqrt{F}$, and therefore $\widetilde{X}_1$ is pushed forward, by the tangent map f_*, to a vector field on the interval $0 < r < 1$,

$$(f_*)_C(\widetilde{X}_1) = 2\sqrt{1 - r^2}\,\frac{\partial}{\partial r}. \tag{3.39}$$

Now, the magnitude of $\widetilde{X}_1$ with respect to the Riemannian metric on M is given by

$$\langle \widetilde{X}_1, \widetilde{X}_1 \rangle_C = \langle UX_1V^*, UX_1V^* \rangle_C = \langle X_1, X_1 \rangle_{C_0} = 1. \tag{3.40}$$

A metric $\mathrm{d}\sigma^2$ is defined on the interval $(0, 1) \cong G\backslash M_1$ through

$$\mathrm{d}\sigma^2((f_*)_C(\widetilde{X}_1), (f_*)_C(\widetilde{X}_1)) = \langle \widetilde{X}_1, \widetilde{X}_1 \rangle_C = 1. \tag{3.41}$$

Equations (3.39) and (3.41) are put together to provide

$$\mathrm{d}\sigma^2\left(2\sqrt{1 - r^2}\,\frac{\partial}{\partial r}, 2\sqrt{1 - r^2}\,\frac{\partial}{\partial r}\right) = 4(1 - r^2)\,\mathrm{d}\sigma^2\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1, \tag{3.42}$$

which shows that the metric $\mathrm{d}\sigma^2$ is expressed as

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{4(1 - r^2)}. \tag{3.43}$$

The length of an interval $[r_1, r_2]$ with $0 < r_1 < r_2 < 1$ is then given by

$$\frac{1}{2}\int_{r_1}^{r_2} \frac{\mathrm{d}r}{\sqrt{1-r^2}} = \frac{1}{2}(\arcsin r_2 - \arcsin r_1). \tag{3.44}$$

Letting r_1 tend to 0, we find that 0 and r is distant by $\frac{1}{2}\arcsin r$, which implies that the state $C \in M$ with concurrence $r = f(C)$ is distant by $\frac{1}{2}\arcsin r$ from the separable states with respect to the Riemannian metric on M.

3.5. A summary of section 3

Proposition 1. *With respect to the bipartite partition, $(\mathbf{C}^2)^{\otimes 3} \cong \mathbf{C}^2 \otimes \mathbf{C}^4 \cong \mathbf{C}^{2\times 4}$, of the three-qubit, the state space M is stratified into three strata, according to the orbit types for the $G = U(2) \times U(4)$ action. Two of the strata are the sets of separable states and maximally entangled states and the other the principal stratum. The sets of separable states and maximally entangled states are diffeomorphic with $S^3 \times_{U(1)} S^7$ and with $V_2(\mathbf{C}^4)$, respectively, and the principal stratum $\dot{M}$ $(= M_1)$ is the direct product of the interval $(0, 1)$ and $U(2) \times_{U(1)\times U(1)} V_2(\mathbf{C}^4)$. The metric defined on the factor space $G\backslash\dot{M} \cong (0, 1)$ is given by (3.43) in terms of the concurrence $r = 2\sqrt{\det(CC^*)}$ with $C \in M \subset \mathbf{C}^{2\times 4}$. It then turns out that the state C with concurrence r is distant by $\frac{1}{2}\arcsin r$ from the separable states.*

4. Four-qubit systems: I

For a four-qubit system $(\mathbf{C}^2)^{\otimes 4}$, there are two inequivalent isomorphisms, $(\mathbf{C}^2)^{\otimes 4} \cong \mathbf{C}^4 \otimes \mathbf{C}^4$ and $(\mathbf{C}^2)^{\otimes 4} \cong \mathbf{C}^2 \otimes \mathbf{C}^8$, with which sections 4 and 5 are concerned, respectively.

4.1. Isotropy subgroups

We study the isotropy subgroup of $G = U(4) \times U(4)$ acting on the state space M given in (2.5) with $(\ell, m) = (2, 2)$. The singular decomposition of $C \in M \subset \mathbf{C}^{4\times 4}$ takes the form $C = UC_0V^*$, where $C_0 = \mathrm{diag}(\mu_1, \ldots, \mu_4)$ and $(U, V) \in U(4) \times U(4)$. Since the isotropy subgroups at C_0 and C are isomorphic to each other, the types of isotropy subgroups are determined at C_0. Our task is then to solve the equation

$$gC_0h^T = C_0, \qquad C_0 = \mathrm{diag}(\mu_1, \ldots, \mu_4), \qquad (g, h) \in U(4) \times U(4), \tag{4.1}$$

according to the types of the diagonal matrix C_0.

(1) $\mu_1 > \mu_2 > \mu_3 > \mu_4 > 0$
Solutions to (4.1) are expressed as

$$g = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_4}), \qquad h = g^{-1} = \mathrm{diag}(\mathrm{e}^{-\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{-\mathrm{i}\theta_4}), \tag{4.2}$$

and the isotropy subgroup proves to be

$$G_{C_0} \cong U(1) \times U(1) \times U(1) \times U(1). \tag{4.3}$$

(2) $\mu_1 > \mu_2 > \mu_3 > \mu_4 = 0$
Solutions to (4.1) are put in the form

$$g = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \mathrm{e}^{\mathrm{i}\theta_3}, \mathrm{e}^{\mathrm{i}\theta_4}), \qquad h = \mathrm{diag}(\mathrm{e}^{-\mathrm{i}\theta_1}, \mathrm{e}^{-\mathrm{i}\theta_2}, \mathrm{e}^{-\mathrm{i}\theta_3}, \mathrm{e}^{\mathrm{i}\theta_5}), \tag{4.4}$$

so that

$$G_{C_0} \cong U(1) \times U(1) \times U(1) \times U(1) \times U(1). \tag{4.5}$$

(3) $\mu_1 = \mu_2 > \mu_3 > \mu_4 > 0$, $\mu_1 > \mu_2 = \mu_3 > \mu_4 > 0$, $\mu_1 > \mu_2 > \mu_3 = \mu_4 > 0$
In the case of $\mu_1 > \mu_2 > \mu_3 = \mu_4 > 0$, solutions to (4.1) are given by

$$g = \begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & k \end{pmatrix}, \qquad h = \begin{pmatrix} e^{-i\theta_1} & & \\ & e^{-i\theta_2} & \\ & & \bar{k} \end{pmatrix}, \qquad k \in U(2), \tag{4.6}$$

which implies, in the respective cases in question, that

$$G_{C_0} \cong U(1) \times U(1) \times U(2). \tag{4.7}$$

(4) $\mu_1 > \mu_2 > \mu_3 = \mu_4 = 0$
Solutions to (4.1) are put in the form

$$g = \begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & k_1 \end{pmatrix}, \qquad h = \begin{pmatrix} e^{-i\theta_1} & & \\ & e^{-i\theta_2} & \\ & & k_2 \end{pmatrix}, \qquad k_1, k_2 \in U(2), \tag{4.8}$$

which implies that

$$G_{C_0} \cong U(1) \times U(1) \times U(2) \times U(2). \tag{4.9}$$

(5) $\mu_1 = \mu_2 > \mu_3 > \mu_4 = 0$, $\mu_1 > \mu_2 = \mu_3 > \mu_4 = 0$
We take the case of $\mu_1 = \mu_2 > \mu_3 > \mu_4 = 0$. Solutions to (4.1) are of the form

$$g = \begin{pmatrix} k & & \\ & e^{i\theta_1} & \\ & & e^{i\theta_2} \end{pmatrix}, \qquad h = \begin{pmatrix} \bar{k} & & \\ & e^{-i\theta_1} & \\ & & e^{i\theta_3} \end{pmatrix}, \qquad k \in U(2), \tag{4.10}$$

and hence, in the present two cases, one obtains

$$G_{C_0} \cong U(2) \times U(1) \times U(1) \times U(1). \tag{4.11}$$

(6) $\mu_1 = \mu_2 > \mu_3 = \mu_4 > 0$
Solutions to (4.1) take the form

$$g = \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix}, \qquad h = \begin{pmatrix} \bar{k}_1 & \\ & \bar{k}_2 \end{pmatrix}, \qquad k_1, k_2 \in U(2), \tag{4.12}$$

so that

$$G_{C_0} \cong U(2) \times U(2). \tag{4.13}$$

(7) $\mu_1 = \mu_2 > \mu_3 = \mu_4 = 0$
Solutions to (4.1) are written as

$$g = \begin{pmatrix} k & \\ & g_2 \end{pmatrix}, \qquad h = \begin{pmatrix} \bar{k} & \\ & h_2 \end{pmatrix}, \qquad k, g_2, h_2 \in U(2), \tag{4.14}$$

which implies that

$$G_{C_0} \cong U(2) \times U(2) \times U(2). \tag{4.15}$$

(8) $\mu_1 = \mu_2 = \mu_3 > \mu_4 > 0$, $\mu_1 > \mu_2 = \mu_3 = \mu_4 > 0$
In the case of $\mu_1 > \mu_2 = \mu_3 = \mu_4 > 0$, solutions to (4.1) are of the form

$$g = \begin{pmatrix} e^{i\theta} & \\ & k_3 \end{pmatrix}, \qquad h = \begin{pmatrix} e^{-i\theta} & \\ & \bar{k}_3 \end{pmatrix}, \qquad k_3 \in U(3), \tag{4.16}$$

and the isotropy subgroup in each case under consideration proves to be

$$G_{C_0} \cong U(1) \times U(3). \tag{4.17}$$

(9) $\mu_1 = \mu_2 = \mu_3 > \mu_4 = 0$
Solutions to (4.1) takes the form

$$g = \begin{pmatrix} k_3 & \\ & \mathrm{e}^{\mathrm{i}\theta_1} \end{pmatrix}, \qquad h = \begin{pmatrix} \overline{k}_3 & \\ & \mathrm{e}^{\mathrm{i}\theta_2} \end{pmatrix}, \qquad k_3 \in U(3), \tag{4.18}$$

so that

$$G_{C_0} \cong U(3) \times U(1) \times U(1). \tag{4.19}$$

(10) $\mu_1 > \mu_2 = \mu_3 = \mu_4 = 0$
Solutions to (4.1) turn out to be

$$g = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & \\ & g_3 \end{pmatrix}, \qquad h = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & \\ & h_3 \end{pmatrix}, \qquad g_3, h_3 \in U(3), \tag{4.20}$$

which provide

$$G_{C_0} \cong U(1) \times U(3) \times U(3). \tag{4.21}$$

(11) $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1/2$
As is easily seen, the isotropy subgroup is

$$G_{C_0} \cong U(4). \tag{4.22}$$

4.2. Orbits

According to the types of isotropy subgroups listed in section 4.1, the state space M is stratified into strata on which orbits are of the same type. There are as many orbit types as those of the isotropy subgroups. Since orbits $\mathcal{O}_C \cong G/G_C$ and $\mathcal{O}_{C_0} \cong G/G_{C_0}$ are diffeomorphic with each other, where $C = UC_0V^*$ with $(U, V) \in U(4) \times U(4)$ and $C_0 = \mathrm{diag}(\mu_1, \ldots, \mu_4)$, we treat $\mathcal{O}_{C_0}$ in what follows. To describe $\mathcal{O}_{C_0}$, we occasionally put the unitary matrices $g, h \in U(4)$ in the form $g = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_4)$ and $h = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_4)$, respectively, where $\boldsymbol{u}_j \cdot \boldsymbol{u}_k = \delta_{jk}$ and $\boldsymbol{v}_j \cdot \boldsymbol{v}_k = \delta_{jk}$.

(1) $\mu_1 > \mu_2 > \mu_3 > \mu_4 > 0$
The orbit through C_0 is expressed as gC_0h^T. Since $gDC_0(h\overline{D})^T = gC_0h^T$ for $D = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_4}) \in U(1)^{\times 4}$, an equivalence relation is defined on $U(4) \times U(4)$ by

$$(gD, h\overline{D}) \sim (g, h), \qquad D \in U(1)^{\times 4}. \tag{4.23}$$

This implies that the orbit through C_0 is described as a factor space,

$$U(4) \times_{U(1)^{\times 4}} U(4), \tag{4.24}$$

which is a fiber bundle over a flag manifold $U(4)/U(1)^{\times 4}$ with fiber $U(4)$, where $U(1)^{\times 4}$ is the maximal torus of $U(4)$.

(2) $\mu_1 > \mu_2 > \mu_3 > \mu_4 = 0$
The orbit through C_0 is expressed as $gC_0h^T = \mu_1\boldsymbol{u}_1\boldsymbol{v}_1^T + \mu_2\boldsymbol{u}_2\boldsymbol{v}_2^T + \mu_3\boldsymbol{u}_3\boldsymbol{v}_3^T$. Since each $\boldsymbol{u}_k\boldsymbol{v}_k^T$ is invariant by the $U(1)$ action (i.e., $\boldsymbol{u}_k\,\mathrm{e}^{\mathrm{i}\theta_k}\,\mathrm{e}^{-\mathrm{i}\theta_k}\boldsymbol{v}_k^T = \boldsymbol{u}_k\boldsymbol{v}_k^T$), an equivalence relation is defined on $V_3(\mathbf{C}^4) \times V_3(\mathbf{C}^4)$ through

$$(U_3D_3, V_3\overline{D}_3) \sim (U_3, V_3), \tag{4.25}$$

where $U_3 = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$, $V_3 = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ and $D_3 = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \mathrm{e}^{\mathrm{i}\theta_3})$, and $V_3(\mathbf{C}^4)$ denotes the Stiefel manifold of orthonormal three frames in $\mathbf{C}^4$. Thus, the orbit is expressed as

$$V_3(\mathbf{C}^4) \times_{U(1)^{\times 3}} V_3(\mathbf{C}^4), \tag{4.26}$$

which is a fiber bundle over the manifold $V_3(\mathbf{C}^4)/U(1)^{\times 3}$ with fiber $V_3(\mathbf{C}^4)$.

(3) $\mu_1 = \mu_2 > \mu_3 > \mu_4 > 0, \mu_1 > \mu_2 = \mu_3 > \mu_4 > 0, \mu_1 > \mu_2 > \mu_3 = \mu_4 > 0$
In the case of $\mu_1 > \mu_2 > \mu_3 = \mu_4 > 0$, the orbit through C_0 is put in the form $gC_0h^T = \mu_1 \boldsymbol{u}_1\boldsymbol{v}_1^T + \mu_2\boldsymbol{u}_2\boldsymbol{v}_2^T + U_2V_2^T$, where $U_2 = (\boldsymbol{u}_3, \boldsymbol{u}_4)$, $V_2 = (\boldsymbol{v}_3, \boldsymbol{v}_4)$. Since $\boldsymbol{u}_1\,\mathrm{e}^{\mathrm{i}\theta_1}(\boldsymbol{v}_1\,\mathrm{e}^{-\mathrm{i}\theta_1})^T = \boldsymbol{u}_1\boldsymbol{v}_1^T$ and since $U_2k(V_2\overline{k})^T = U_2V_2^T$ for $k \in U(2)$, an equivalence relation is defined on $U(4) \times U(4)$ through

$$(\boldsymbol{u}_1, \boldsymbol{u}_2, U_2, \boldsymbol{v}_1, \boldsymbol{v}_2, V_2) \sim (\boldsymbol{u}_1\,\mathrm{e}^{\mathrm{i}\theta_1}, \boldsymbol{u}_2\,\mathrm{e}^{\mathrm{i}\theta_2}, U_2k, \boldsymbol{v}_1\,\mathrm{e}^{-\mathrm{i}\theta_1}, \boldsymbol{v}_2\,\mathrm{e}^{-\mathrm{i}\theta_2}, V_2\overline{k}), \tag{4.27}$$

where $(\boldsymbol{u}_1, \boldsymbol{u}_2, U_2) \in U(4)$ and $(\boldsymbol{v}_1, \boldsymbol{v}_2, V_2) \in U(4)$. Hence, in the three cases in question, the orbit space is expressed as

$$U(4) \times_{U(1)\times U(1)\times U(2)} U(4), \tag{4.28}$$

which is a fiber bundle over the manifold $U(4)/(U(1)\times U(1)\times U(2)) \cong V_2(\mathbf{C}^4)/(U(1)\times U(1))$ with fiber $U(4)$.

(4) $\mu_1 > \mu_2 > \mu_3 = \mu_4 = 0$
The orbit through C_0 is expressed as $gC_0h^T = \mu_1\boldsymbol{u}_1\boldsymbol{v}_1^T + \mu_2\boldsymbol{u}_2\boldsymbol{v}_2^T$, which determines an equivalence relation on $V_2(\mathbf{C}^4) \times V_2(\mathbf{C}^4)$ through

$$(\boldsymbol{u}_1\,\mathrm{e}^{\mathrm{i}\theta_1}, \boldsymbol{u}_2\,\mathrm{e}^{\mathrm{i}\theta_2}, \boldsymbol{v}_1\,\mathrm{e}^{-\mathrm{i}\theta_1}, \boldsymbol{v}_2\,\mathrm{e}^{-\mathrm{i}\theta_2}) \sim (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2). \tag{4.29}$$

Hence, the orbit is diffeomorphic with

$$V_2(\mathbf{C}^4) \times_{U(1)\times U(1)} V_2(\mathbf{C}^4), \tag{4.30}$$

which is a fiber bundle over the manifold $V_2(\mathbf{C}^4)/(U(1) \times U(1))$ with fiber $V_2(\mathbf{C}^4)$.

(5) $\mu_1 = \mu_2 > \mu_3 > \mu_4 = 0, \mu_1 > \mu_2 = \mu_3 > \mu_4 = 0$
We take the case of $\mu_1 = \mu_2 > \mu_3 > \mu_4 = 0$. Then, the orbit through C_0 is expressed as $gC_0h^T = \mu_1U_2V_2^T + \mu_3\boldsymbol{u}_3\boldsymbol{v}_3^T$, where $U_2 = (\boldsymbol{u}_1, \boldsymbol{u}_2)$, $V_2 = (\boldsymbol{v}_1, \boldsymbol{v}_2)$. This determines an equivalence relation on $V_3(\mathbf{C}^4) \times V_3(\mathbf{C}^4)$ through

$$(U_2k, \boldsymbol{u}_3\,\mathrm{e}^{\mathrm{i}\theta}, V_2\overline{k}, \boldsymbol{v}_3\,\mathrm{e}^{-\mathrm{i}\theta}) \sim (U_2, \boldsymbol{u}_3, V_2, \boldsymbol{v}_3), \quad (k, \mathrm{e}^{\mathrm{i}\theta}) \in U(2) \times U(1), \tag{4.31}$$

and hence the orbit is expressed, in each case in question, as

$$V_3(\mathbf{C}^4) \times_{U(2)\times U(1)} V_3(\mathbf{C}^4), \tag{4.32}$$

which is a fiber bundle over $V_3(\mathbf{C}^4)/(U(2) \times U(1))$ with fiber $V_3(\mathbf{C}^4)$.

(6) $\mu_1 = \mu_2 > \mu_3 = \mu_4 > 0$
The orbit through C_0 takes the form $gC_0h^T = \mu_1U_2V_2^T + \mu_3U_2'(V_3')^T$, where $U_2 = (\boldsymbol{u}_1, \boldsymbol{u}_2)$, $U_2' = (\boldsymbol{u}_3, \boldsymbol{u}_4)$, $V_2 = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ and $V_2' = (\boldsymbol{v}_3, \boldsymbol{v}_4)$. Hence, an equivalence relation is defined on $U(4) \times U(4)$ through

$$(U_2k_1, U_2'k_2, V_2\overline{k}_1, V_2'\overline{k}_2) \sim (U_2, U_2', V_2, V_2'), \quad (k_1, k_2) \in U(2) \times U(2), \tag{4.33}$$

so that the orbit is put in the form

$$U(4) \times_{U(2)\times U(2)} U(4), \tag{4.34}$$

which is a fiber bundle over the Grassmann manifold $U(4)/(U(2) \times U(2)) \cong G_2(\mathbf{C}^4)$ with fiber $U(4)$.

(7) $\mu_1 = \mu_2 > \mu_3 = \mu_4 = 0$
The orbit through C_0 takes the form $gC_0h^T = \mu_1U_2V_2^T$, which determines an equivalence relation on $V_2(\mathbf{C}^4) \times V_2(\mathbf{C}^4)$ through

$$(U_2k, V_2\overline{k}) \sim (U_2, V_2), \qquad k \in U(2). \tag{4.35}$$

Hence, we have the orbit

$$V_2(\mathbf{C}^4) \times_{U(2)} V_2(\mathbf{C}^4), \tag{4.36}$$

which is a fiber bundle over the Grassmann manifold $V_2(\mathbf{C}^4)/U(2) \cong G_2(\mathbf{C}^4)$ with fiber $V_2(\mathbf{C}^4)$.

(8) $\mu_1 = \mu_2 = \mu_3 > \mu_4 > 0, \mu_1 > \mu_2 = \mu_3 = \mu_4 > 0$
We treat the case of $\mu_1 = \mu_2 = \mu_3 > \mu_4 > 0$. The orbit through C_0 is given by $gC_0h^T = \mu_1 U_3 V_3^T + \mu_4 \boldsymbol{u}_4 \boldsymbol{v}_4^T$, where $U_3 = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$ and $V_3 = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$. Then, an equivalence relation is defined on $U(4) \times U(4)$ through

$$(U_3, \boldsymbol{u}_4, V_3, \boldsymbol{v}_4) \sim (U_3 h, \boldsymbol{u}_4 \,\mathrm{e}^{\mathrm{i}\theta}, V_3 \overline{h}, \boldsymbol{v}_4 \,\mathrm{e}^{-\mathrm{i}\theta}), \qquad (h, \mathrm{e}^{\mathrm{i}\theta}) \in U(3) \times U(1), \tag{4.37}$$

where $(U_3, \boldsymbol{u}_4) \in U(4)$ and $(V_3, \boldsymbol{v}_4) \in U(4)$. In each case in question, the orbit is expressed as

$$U(4) \times_{U(3)\times U(1)} U(4), \tag{4.38}$$

which is a fiber bundle over the Grassmann manifold $U(4)/(U(3) \times U(1)) \cong G_3(\mathbf{C}^4)$ with fiber $U(4)$.

(9) $\mu_1 = \mu_2 = \mu_3 > \mu_4 = 0$
We have the orbit of the form $gC_0h^T = \mu_1 U_3 V_3^T$. An equivalence relation is then defined on $V_3(\mathbf{C}^4) \times V_3(\mathbf{C}^4)$ through

$$(U_3 k_3, V_3 \overline{k}_3) \sim (U_3, V_3), \qquad k_3 \in U(3), \tag{4.39}$$

so that the orbit is described as

$$V_3(\mathbf{C}^4) \times_{U(3)} V_3(\mathbf{C}^4), \tag{4.40}$$

which is a fiber bundle over the Grassmann manifold $V_3(\mathbf{C}^4)/U(3) \cong G_3(\mathbf{C}^4)$ with fiber $V_3(\mathbf{C}^4)$.

(10) $\mu_1 > \mu_2 = \mu_3 = \mu_4 = 0$
The orbit through $C_0 = \mathrm{diag}(1, 0, 0, 0)$ proves to be $gC_0h^T = \boldsymbol{u}_1 \boldsymbol{v}_1^T$. Since $|\boldsymbol{u}_1| = |\boldsymbol{v}_1| = 1$ and since $\boldsymbol{u}_1 \,\mathrm{e}^{\mathrm{i}\theta} \mathrm{e}^{-\mathrm{i}\theta} \boldsymbol{v}_1^T = \boldsymbol{u}_1 \boldsymbol{v}_1^T$, we have the equivalence relation

$$(\boldsymbol{u}_1, \boldsymbol{v}_1) \sim (\boldsymbol{u}_1 \,\mathrm{e}^{\mathrm{i}\theta}, \boldsymbol{v}_1 \,\mathrm{e}^{-\mathrm{i}\theta}) \tag{4.41}$$

on the product space $S^7 \times S^7$. The orbit through C_0 is then diffeomorphic with

$$S^7 \times_{U(1)} S^7, \tag{4.42}$$

which is a fiber bundle over $S^7/U(1) \cong \mathbf{C}P^3$ with fiber S^7.

(11) $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1/2$
The orbit through $C_0 = \frac{1}{2} I$ is the set of all $gC_0h^T = \frac{1}{2} gh^T$ with $(g, h) \in U(4) \times U(4)$, which is diffeomorphic with $U(4)$. The orbit is then

$$U(4) \times_{U(4)} U(4) \cong U(4). \tag{4.43}$$

Among those orbits obtained above, we have found the sets of separable states and maximally entangled states with respect to the isomorphism $(\mathbf{C}^2)^{\otimes 4} \cong \mathbf{C}^4 \otimes \mathbf{C}^4$, which are diffeomorphic with $S^7 \times_{U(1)} S^7$ and $U(4)$, respectively.

4.3. The tangent space decomposition

The dimension of the vertical subspace $V_C \cong \mathcal{G}/\mathcal{G}_C$ depends on the types of the isotropy subgroups G_C. Since we have already obtained the isotropy subgroups, we can easily discuss $V_C \cong \mathcal{G}/\mathcal{G}_C$, but we will not look into V_C.

We turn to the horizontal subspaces H_C. If C and $C_0 = \mathrm{diag}(\mu_1, \ldots, \mu_4)$ are sitting on the same orbit of G, H_C and H_{C_0} are isomorphic to each other. In view of this, we find H_{C_0} by solving the equations

$$A^* C_0 = C_0 A, \qquad A C_0 = C_0 A^*, \qquad A \in \mathbf{C}^{4\times 4}. \tag{4.44}$$

These equations are written componentwise as $\overline{a}_{kj}\mu_k = \mu_j a_{jk}$ and $a_{jk}\mu_k = \mu_j\overline{a}_{kj}$. We will give solutions only for several types of singular values.

If $\mu_1 > \mu_2 > \mu_3 > \mu_4 > 0$, solutions to (4.44) prove to be

$$A = \mathrm{diag}(a_1, \dots, a_4), \qquad \overline{a}_k = a_k, k = 1, \dots, 4. \tag{4.45}$$

Since A is tangent to M at C_0, A should satisfy the condition $\mathrm{tr}(A^*C_0 + C_0^*A) = 0$, so that a_i should be subject to

$$\sum_{k=1}^{4} \mu_k a_k = 0. \tag{4.46}$$

There are three linearly independent solutions satisfying the above condition, and therefore a basis of H_{C_0} is given, for example, by

$$X_1 = \mathrm{diag}(\mu_2, -\mu_1, \mu_4, -\mu_3), \tag{4.47}$$

$$X_2 = \mathrm{diag}(\mu_3, -\mu_4, -\mu_1, \mu_2), \tag{4.48}$$

$$X_3 = \mathrm{diag}(\mu_4, \mu_3, -\mu_2, -\mu_1). \tag{4.49}$$

As is easily verified, these vectors are orthogonal to one another with respect to the Riemannian metric on M:

$$\langle X_k, X_j\rangle_{C_0} = \tfrac{1}{2}\,\mathrm{tr}(X_k^*X_j + X_j^*X_k) = \delta_{kj}. \tag{4.50}$$

In the case of $\mu_1 > \mu_2 > \mu_3 > \mu_4 = 0$, we obtain, in place of (4.45),

$$A = \mathrm{diag}(a_1, a_2, a_3, a_4 + i a_5), \qquad \overline{a}_j = a_j, \quad j = 1, \dots, 4, 5. \tag{4.51}$$

The tangential condition $\mathrm{tr}(A^*C_0 + C_0^*A) = 0$ results in the same equation as (4.46). In addition to (4.47)–(4.49), we have another horizontal vector independent of them,

$$X_4 = \mathrm{diag}(0, 0, 0, \mathrm{i}), \qquad \mathrm{i} = \sqrt{-1}. \tag{4.52}$$

The set of vectors X_1, X_2, X_3, X_4 forms a basis of H_{C_0} at $C_0 = \mathrm{diag}(\mu_1, \mu_2, \mu_3, 0)$ with $\mu_1 > \mu_2 > \mu_3 > 0$.

In the cases of $\mu_1 > \mu_2 > \mu_3 = \mu_4 > 0$ and $\mu_1 > \mu_2 = \mu_3 = \mu_4 > 0$, the condition $a_{jk}\mu_k = \mu_j\overline{a}_{kj}$ implies that $a_{jj} = \overline{a}_{jj}$ for $j = 1, 2$ and $a_{jk} = \overline{a}_{kj}$ for $j, k \in \{3, 4\}$ and that $a_{jj} = \overline{a}_{jj}$ for $j = 1$ and $a_{jk} = \overline{a}_{kj}$ for $j, k \in \{2, 3, 4\}$, respectively. In both cases, the condition $\mathrm{tr}(A^*C_0 + C_0^*A) = 0$ provides $\sum a_{kk}\mu_k = 0$. Hence, the horizontal vectors are, respectively, expressed as

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & a_{34} \\ & & \overline{a}_{34} & a_4 \end{pmatrix}, \qquad a_k = \overline{a}_k, \qquad \sum_{k=1}^{4} a_k\mu_k = 0, \qquad \mu_1 > \mu_2 > \mu_3 = \mu_4 > 0, \tag{4.53}$$

and as

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & a_{23} & a_{24} \\ & \overline{a}_{23} & a_3 & a_{34} \\ & \overline{a}_{24} & \overline{a}_{34} & a_4 \end{pmatrix}, \qquad a_k = \overline{a}_k, \qquad \sum_{k=1}^{4} a_k\mu_k = 0, \qquad \mu_1 > \mu_2 = \mu_3 = \mu_4 > 0. \tag{4.54}$$

4.4. Measurement of entanglement

According to the orbit types discussed in section 4.2, M is stratified into strata. The set of C such that all the singular values $\mu_k(C)$ are positive and distinct is a principal stratum, which we denote by $\dot{M}$. The factor space $G\backslash\dot{M}$ will prove to be diffeomorphic with

$$P_3 := \left\{ \boldsymbol{x} = (x_k) \in \mathbf{R}^4 \,\middle|\, \sum_{k=1}^{4} x_k^2 = 1, x_1 > \cdots > x_4 > 0 \right\}, \tag{4.55}$$

which is an open submanifold of the sphere $S^3 \subset \mathbf{R}^4$. The projection map is given by

$$\pi_3 : \dot{M} \to G\backslash\dot{M} \cong P_3; C \mapsto \boldsymbol{\mu}(C) = (\mu_k(C)), \tag{4.56}$$

where $\boldsymbol{\mu}(C)$ denote the vector consisting of the singular values of C with $\mu_1(C) > \mu_2(C) > \mu_3(C) > \mu_4(C) > 0$. As is easily seen, this map is surjective onto P_3. We now assume that $\boldsymbol{\mu}(C_1) = \boldsymbol{\mu}(C_2)$ for $C_1, C_2 \in \dot{M}$. Then, there exists $(g_1, h_1), (g_2, h_2) \in U(4)\times U(4)$ such that $C_1 = g_1 C_0 h_1^T$, $C_2 = g_2 C_0 h_2^T$, where C_0 is the diagonal matrix with singular values as diagonal elements. These equations imply that C_1 and C_2 lie in the same orbit of $G = U(4) \times U(4)$ through C_0. Hence, the inverse image of $p \in P_3$ is an orbit of G, which implies that $G\backslash\dot{M} \cong P_3$.

We are interested in a metric to be defined on $G\backslash\dot{M}$. From (4.45) and (4.46), the horizontal subspace H_{C_0} at $C_0 = \mathrm{diag}(\mu_1, \ldots, \mu_4)$ with $\mu_1 > \cdots > \mu_4 > 0$ and $\sum_{j=1}^4 \mu_j^2 = 1$ is given by

$$H_{C_0} = \left\{ \mathrm{diag}(a_1, \ldots, a_4) \,\middle|\, a_j \in \mathbf{R}, \sum_{j=1}^{4} \mu_j a_j = 0 \right\}. \tag{4.57}$$

Since the horizontal subspace $H_C \cong H_{C_0}$ projects isomorphically onto the tangent space to the factor space $G\backslash\dot{M}$ at $\pi_3(C)$, and since the metric on $\dot{M}$ is invariant under the action of $G = U(4) \times U(4)$, the metric on $\dot{M}$ projects to that on $G\backslash\dot{M}$ through

$$\mathrm{d}\sigma^2((\pi_{3*})_C(\widetilde{X}_j), (\pi_{3*})_C(\widetilde{X}_k)) = \langle \widetilde{X}_j, \widetilde{X}_k \rangle_C = \langle X_j, X_k \rangle_{C_0} = \delta_{jk}, \tag{4.58}$$

where $\widetilde{X}_j = U X_j V^*$ with $C = U C_0 V^*$ and X_j are tangent vectors given in (4.47)–(4.49). In view of (4.57) and (4.58), the metric $\mathrm{d}\sigma^2$ is also written as

$$\mathrm{d}\sigma^2 = \mathrm{tr}(\mathrm{d}C_0\,\mathrm{d}C_0^*) = \sum_{j=1}^{4} \mathrm{d}\mu_j^2, \qquad \frac{1}{2}\mathrm{tr}(\mathrm{d}C_0 C_0^* + C_0 \mathrm{d}C_0^*) = \sum_{j=1}^{4} \mu_j\,\mathrm{d}\mu_j = 0, \tag{4.59}$$

where $\mathrm{d}C_0 = \mathrm{diag}(\mathrm{d}\mu_1, \ldots, \mathrm{d}\mu_4)$. In other words, the metric $\mathrm{d}\sigma^2$ defined on $G\backslash\dot{M}$ is nothing but the metric on P_3 induced from the standard metric on S^3. We note that the metric on P_3 can be extended to the closure $\overline{P}_3$ of P_3.

As was stated in the introduction, the function $F(C) = \det(I - CC^*)$ serves as a measure of entanglement. As is easily verified, the range of F is $0 \leqslant F(C) \leqslant (3/4)^4$. In particular, for the separable and maximally entangled states, we have $F(C) = 0$ and $F(C) = (3/4)^4$, respectively. In view of this, we may define the concurrence of C to be $f(C) = (4/3)F(C)^{1/4}$. Then, the range of f becomes $0 \leqslant f(C) \leqslant 1$. However, we work with $F(C)$ for simplicity.

Since F is invariant under the G action, it projects to a function $\widetilde{F}$ on $\overline{P}_3$ through $\widetilde{F}(\boldsymbol{\mu}(C)) = F(C)$. The sets of separable states and maximally entangled states are mapped to $\boldsymbol{e}_1 = (1, 0, 0, 0)^T$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$ by π_3, respectively, which are located in the boundary of P_3. We wish to measure the distance between $\boldsymbol{e}_1$ and the level set $\widetilde{F}^{-1}(k)$ with $0 \leqslant k \leqslant (3/4)^4$. Since the metric on $\overline{P}_3 \subset S^3$ is the standard one, the distance between $\boldsymbol{e}_1$ and $\boldsymbol{x} \in \overline{P}_3$ is given by $\arccos(\boldsymbol{e}_1 \cdot \boldsymbol{x}) = \arccos x_1$, so that the distance in question is determined by minimizing $\arccos x_1$. Put another way, our problem is reduced to the following:

$$\text{maximize } x_1, \text{ subject to } \sum_{j=1}^{4} x_j^2 = 1, \qquad \widetilde{F}(\boldsymbol{x}) = k, \tag{4.60}$$

where we have not taken into account the restriction $x_1 \geqslant x_2 \geqslant x_3 \geqslant x_4 \geqslant 0$, which will be considered soon after. Introducing Lagrange's multipliers λ and μ, we work with the function

$$x_1 + \lambda \left(\sum_{j=1}^{4} x_j^2 - 1 \right) + \mu\left(\left(1 - x_1^2\right) \cdots \left(1 - x_4^2\right) - k\right). \tag{4.61}$$

The condition for this function to take an extremum is easy to write after differentiation, from which we can find that $x_2 = x_3 = x_4$, a condition subject to the restriction stated above. When this occurs, one has $x_1^2 + \cdots + x_4^2 = x_1^2 + 3x_2^2 = 1$, so that

$$k = \left(1 - x_1^2\right)\left(1 - x_2^2\right)^3 = 3x_2^2\left(1 - x_2^2\right)^3, \tag{4.62}$$

which determines x_2 as a function of k. There are two solutions, $x_2^+(k)$ and $x_2^-(k)$, to the above equation in general, where $x_2^+(k)$ is greater than or equal to $1/2$ and $x_2^-(k)$ less than or equal to $1/2$. For these values, we have $x_1^{\pm}(k) = \sqrt{1 - 3\left(x_2^{\pm}(k)\right)^2}$. This implies that $x_2^-(k)$ yields the maximum, $x_1^-(k)$, of x_1, so that the distance is given by

$$\arccos \sqrt{1 - 3(x_2^-(k))^2}, \tag{4.63}$$

where $x_2^-(k)$ denotes the smaller one of solutions to (4.62). In particular, if $x_2^-(k) = 1/2$ with $k = (3/4)^4$, we have the distance $\arccos \sqrt{1 - 3(x_2^-(k))^2} = \arccos(1/2) = \pi/3$, which is the distance between $\boldsymbol{e}_1$ and $\boldsymbol{x} = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)^T$. In other words, the distance between the separable states and the maximally entangled states is $\pi/3$ with respect to the canonical Riemannian metric on the state space.

4.5. A summary of section 4

Proposition 2. *With respect to the bipartite partition, $(\mathbf{C}^2)^{\otimes 4} \cong \mathbf{C}^4 \otimes \mathbf{C}^4 \cong \mathbf{C}^{4\times 4}$, of the four-qubit, the state space M is stratified into eleven strata, according to the orbit types of the $G = U(4) \times U(4)$ action, as is listed in section 4.2. Two of the strata are the sets of separable states and maximally entangled states, which are diffeomorphic with $S^7 \times_{U(1)} S^7$ and $U(4)$, respectively. The orbit space $G\backslash\dot{M}$ for the principal stratum $\dot{M}$ is diffeomorphic with an open submanifold P_3 of the sphere S^3, where P_3 is given by (4.55). As is given in (4.59), the metric defined on P_3 through the submersion $\pi_3 : \dot{M} \to P_3$ is isometric with the canonical metric on the sphere S^3 restricted to P_3. Let $F(C)$ be the entanglement measure defined in (2.8) with $C \in M \subset \mathbf{C}^{4\times 4}$. This metric can serve to measure the distance between the level sets $F^{-1}(k)$ and $F^{-1}(0)$, where $0 \leqslant k \leqslant (3/4)^4$ and $F^{-1}(0)$ is the set of separable states. The resultant distance is given in (4.63).*

5. Four-qubit systems: II

5.1. Isotropy subgroups

The group $G = U(2) \times U(8)$ acts on the state space M for a four-qubit system in the manner (2.9) with $(\ell, m) = (1, 3)$. Isotropy subgroups can be found in a similar manner to that in subsection 3.1. Like (3.7), the isotropy subgroups are classified into three types:

$$G_C \cong \begin{cases} U(1) \times U(1) \times U(7), & \text{if} \quad \det(CC^*) = 0, \\ U(1) \times U(1) \times U(6), & \text{if} \quad \det(CC^*) \neq 0, \quad \mu_1(C) > \mu_2(C), \\ U(2) \times U(6), & \text{if} \quad \det(CC^*) \neq 0, \quad \mu_1(C) = \mu_2(C), \end{cases} \tag{5.1}$$

where $\mu_j(C)$ denote the singular values of C.

5.2. Orbits

According to the types of isotropy subgroups listed above, the state space M is stratified into strata. Orbit types are also classified accordingly. As in the preceding cases, we have only to find the orbit through $C_0 = (\Lambda, 0) \in \mathbf{C}^{2\times 8}$, $\Lambda = \mathrm{diag}(\mu_1, \mu_2)$, for each type of Λ. In what follows, we occasionally put the unitary matrices $g \in U(2), h \in U(8)$ in the form $g = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ and $h = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_8)$, respectively, where $\boldsymbol{u}_j \cdot \boldsymbol{u}_k = \delta_{jk}$ and $\boldsymbol{v}_\ell \cdot \boldsymbol{v}_m = \delta_{\ell m}$.

(1) $\mu_1 > \mu_2 > 0$

The orbit through C_0 is expressed as $gC_0h^T = \mu_1\boldsymbol{u}_1\boldsymbol{v}_1^T + \mu_2\boldsymbol{u}_2\boldsymbol{v}_2^T$. Since $\boldsymbol{u}_1\boldsymbol{v}_1^T = \boldsymbol{u}_1\,\mathrm{e}^{\mathrm{i}\theta_1}\,\mathrm{e}^{-\mathrm{i}\theta_1}\boldsymbol{v}_1^T$ and $\boldsymbol{u}_2\boldsymbol{v}_2^T = \boldsymbol{u}_2\,\mathrm{e}^{\mathrm{i}\theta_2}\,\mathrm{e}^{-\mathrm{i}\theta_2}\boldsymbol{v}_2^T$, one can define an equivalence relation on $U(2) \times V_2(\mathbf{C}^8)$ by

$$(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2) \sim (\boldsymbol{u}_1\,\mathrm{e}^{\mathrm{i}\theta_1}, \boldsymbol{u}_2\,\mathrm{e}^{\mathrm{i}\theta_2}, \boldsymbol{v}_1\,\mathrm{e}^{-\mathrm{i}\theta_1}, \boldsymbol{v}_2\,\mathrm{e}^{-\mathrm{i}\theta_2}). \tag{5.2}$$

Thus the orbit space is put in the form

$$U(2) \times_{U(1)\times U(1)} V_2(\mathbf{C}^8), \tag{5.3}$$

which is a fiber bundle over the sphere $S^2 \cong U(2)/(U(1) \times U(1))$ with fiber $V_2(\mathbf{C}^8)$.

(2) $\mu_1 = \mu_2 = 1/\sqrt{2}$

The orbit through C_0 is given by $gC_0h^T = \frac{1}{\sqrt{2}}UV_2^T$, where $U = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ and $V_2 = (\boldsymbol{v}_1, \boldsymbol{v}_2)$. From this, an equivalence relation is defined on $U(2) \times V_2(\mathbf{C}^8)$ through

$$(Uk, V_2\overline{k}) \sim (U, V_2), \qquad k \in U(2). \tag{5.4}$$

Hence, the orbit is put in the form

$$U(2) \times_{U(2)} V_2(\mathbf{C}^8) \cong V_2(\mathbf{C}^8). \tag{5.5}$$

(3) $\mu_1 > \mu_2 = 0$

The orbit is expressed as $gC_0h^T = \boldsymbol{u}_1\boldsymbol{v}_1^T$, which determines an equivalence relation on $S^3 \times S^{15}$ through

$$(\boldsymbol{u}_1\,\mathrm{e}^{\mathrm{i}\theta}, \boldsymbol{v}_1\,\mathrm{e}^{-\mathrm{i}\theta}) \sim (\boldsymbol{u}_1, \boldsymbol{v}_1), \tag{5.6}$$

so that the orbit is written as

$$S^3 \times_{U(1)} S^{15}, \tag{5.7}$$

which is a fiber bundle over the sphere $S^2 \cong S^3/U(1)$ with fiber S^{15}.

5.3. The tangent space decomposition

From the isotropy subgroups obtained in subsection 5.1, the vertical subspaces $V_C \cong \mathcal{G}/\mathcal{G}_C$ are easy to obtain. We will not look into respective types of V_C. We turn to the horizontal subspaces $H_C \cong H_{C_0}$. To find basis vectors of H_{C_0}, we have to solve the equations

$$X^* C_0 = C_0 X, \qquad X C_0 = C_0 X^*, \tag{5.8}$$

where

$$C_0 = (\Lambda, 0), \qquad \Lambda = \mathrm{diag}(\mu_1, \mu_2), \qquad 0 \in \mathbf{C}^{2\times 6}. \tag{5.9}$$

The calculation results in the following: suppose $\det \Lambda \neq 0$, and let

$$A_1 = \begin{pmatrix} \mu_2 & 0 \\ 0 & -\mu_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & -\mathrm{i}\mu_2 \\ \mathrm{i}\mu_1 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & \mu_2 \\ \mu_1 & 0 \end{pmatrix}. \tag{5.10}$$

Then, a basis of horizontal vectors is given by

$$X_1 = (A_1, 0) \in H_{C_0}, \qquad \text{if} \quad \mu_1 > \mu_2 > 0, \tag{5.11a}$$

$$X_k = (A_k, 0) \in H_{C_0}, \quad k = 1, 2, 3, \qquad \text{if} \quad \mu_1 = \mu_2 \neq 0. \tag{5.11b}$$

If $\det \Lambda = 0$, 14 independent vectors are found as in (3.14).

5.4. Measurement of entanglement

Discussion runs in parallel with that in section 4.4. The set of C such that $\mu_1(C) > \mu_2(C) > 0$ is the principal stratum in M, which we denote by $\dot{M}$. The factor space $G\backslash\dot{M}$ proves to be diffeomorphic with

$$P_1 := \left\{ \boldsymbol{x} = (x_j) \in \mathbf{R}^2 \,\middle|\, \sum_{j=1}^{2} x_j^2 = 1, x_1 > x_2 > 0 \right\}, \tag{5.12}$$

which is part of the circle $S^1 \subset \mathbf{R}^2$. The projection map is given by

$$\pi_1 : \dot{M} \to P \cong G\backslash\dot{M};\ C \mapsto \boldsymbol{\mu}(C) = (\mu_j(C)), \tag{5.13}$$

where $\boldsymbol{\mu}(C)$ denote the vector consisting of the singular values of C with $\mu_1(C) > \mu_2(C) > 0$. The proof runs in parallel with that in subsection 4.4.

We are interested in a metric to be defined on $G\backslash\dot{M}$. As is seen already, the horizontal subspace H_{C_0} at $C_0 = (\Lambda, 0)$ with $\Lambda = \mathrm{diag}(\mu_1, \mu_2)$ and $\mu_1 > \mu_2 > 0$ is spanned by X_1 given in (5.11a). If C_0 and C are sitting in the same orbit, H_{C_0} and H_C are isomorphic to each other. Since the horizontal subspace H_C projects isomorphically onto the tangent space to the factor space $G\backslash\dot{M}$ at $\pi_1(C)$, the metric on $\dot{M}$, which is invariant under the $G = U(2) \times U(8)$ action, projects to $G\backslash\dot{M}$, and is expressed, like (4.59), as

$$\mathrm{tr}(\mathrm{d}C_0\,\mathrm{d}C_0^*) = \sum_{j=1}^{2} \mathrm{d}\mu_j^2, \qquad \mathrm{tr}(\mathrm{d}C_0 C_0^* + C_0 \mathrm{d}C_0^*) = \sum_{j=1}^{2} \mu_j\,\mathrm{d}\mu_j = 0. \tag{5.14}$$

In other words, the metric defined on the factor space is nothing but the metric on P_1 induced from the standard metric on S^1. The metric on P_1 can be extended to the boundary of P_1. If we introduce the parameter θ by $x_1 = \cos\theta, x_2 = \sin\theta$ with $0 \leqslant \theta \leqslant \frac{\pi}{4}$, the induced metric is expressed as $\mathrm{d}\theta^2$, which is a trivial fact. We introduce the concurrence r

by $r = 2\sqrt{\det(CC^*)} = 2x_1x_2$, where we have to note that $\det(I - CC^*) = \det(CC^*)$ in this case. Thus, the metric on P_1 turns out to be put in the form

$$\mathrm{d}\theta^2 = \frac{\mathrm{d}r^2}{4(1-r^2)}, \tag{5.15}$$

which is the same as that in (3.43). The distance between the separable states and the states of concurrence r is of course given by $\theta = \frac{1}{2}\arcsin r$.

5.5. A summary of section 5

Proposition 3. *With respect to the bipartite partition, $(\mathbf{C}^2)^{\otimes 4} \cong \mathbf{C}^2 \otimes \mathbf{C}^8 \cong \mathbf{C}^{2\times 8}$, of the four-qubit, the state space M is stratified into three strata, as is listed in section 5.2, according to the $G = U(2) \times U(8)$ action. Two of the strata are the sets of separable states and maximally entangled states, which are diffeomorphic with $S^3 \times_{U(1)} S^{15}$ and $V_2(\mathbf{C}^8)$, respectively. The orbit space $G\backslash\dot{M}$ of the principal stratum $\dot{M}$ is diffeomorphic with an open submanifold P_1 of the circle S^1, where P_1 is given in (5.12). The metric defined on P_1 through the submersion $\pi_1 : \dot{M} \to P_1$ is isometric with the canonical metric on the circle S^1 restricted to P_1. If one introduces the concurrence by $r = 2\sqrt{\det(CC^*)}$, the metric takes the form of (5.15), like (3.43). The distance between the separable states and the states of concurrence r is then given by $\theta = \frac{1}{2}\arcsin r$.*

6. Multi-qubit systems

6.1. Some of isotropy subgroups and orbits

The isotropy subgroups of $G = U(2^\ell) \times U(2^m)$ are determined by the types of singular values. For simplicity, we here take up only three types of singular values such as (i) $\mu_1 > \mu_2 > \cdots > \mu_{2^\ell} > 0$, (ii) $\mu_1 > \mu_2 = \cdots = \mu_{2^\ell} = 0$ and (iii) $\mu_1 = \mu_2 = \cdots = \mu_{2^\ell}$. The singular decomposition of C is put in the form

$$C = UC_0V^*, \qquad (U, V) \in U(2^\ell) \times U(2^m), \tag{6.1}$$

where

$$C_0 = (\Lambda, 0), \qquad \Lambda = \mathrm{diag}(\mu_1, \ldots, \mu_{2^\ell}). \tag{6.2}$$

Like (3.7), we obtain isotropy subgroups

$$G_{C_0} \cong \begin{cases} U(1)^{\times 2^\ell} \times U(2^m - 2^\ell), & \text{(i)} \quad \mu_1 > \mu_2 > \cdots > \mu_{2^\ell} > 0, \\ U(1) \times U(2^\ell - 1) \times U(2^m - 1), & \text{(ii)} \quad \mu_1 > \mu_2 = \cdots = \mu_{2^\ell} = 0, \\ U(2^\ell) \times U(2^m - 2^\ell), & \text{(iii)} \quad \mu_1 = \mu_2 = \cdots = \mu_{2^\ell}. \end{cases} \tag{6.3}$$

According to the above isotropy subgroups, we can obtain the orbits, in the same manner as in subsection 5.2, as follows:

$$\text{(i)} \quad U(2^\ell) \times_{U(1)^{\times 2^\ell}} V_{2^\ell}(\mathbf{C}^{2^m}), \tag{6.4}$$

$$\text{(ii)} \quad S^{2^{\ell+1}-1} \times_{U(1)} S^{2^{m+1}-1}, \tag{6.5}$$

$$\text{(iii)} \quad U(2^\ell) \times_{U(2^\ell)} V_{2^\ell}(\mathbf{C}^{2^m}) \cong V_{2^\ell}(\mathbf{C}^{2^m}). \tag{6.6}$$

Among the above, the second and the last ones are the sets of separable states and maximally entangled states, respectively.

6.2. Measurement of entanglement

We are interested in the principal stratum $\dot{M}$ which is the set of C whose singular values are subject to $\mu_1 > \mu_2 > \cdots > \mu_{2^\ell} > 0$. Then, like (4.55) and (4.56), it turns out that

$$G\backslash\dot{M} \cong P := \left\{ \boldsymbol{x} = (x_j) \in \mathbf{R}^{2^\ell} \,\middle|\, \sum_{j=1}^{2^\ell} x_j^2 = 1,\ x_1 > \cdots > x_{2^\ell} > 0 \right\}, \tag{6.7}$$

$$\pi : \dot{M} \longrightarrow P \cong G\backslash\dot{M};\, C \mapsto \boldsymbol{\mu}(C) = (\mu_j(C)). \tag{6.8}$$

Like (4.57) and (5.11*a*), the horizontal subspace at C_0 is given by

$$H_{C_0} = \left\{ X = (A, 0) \in \mathbf{C}^{2^\ell \times 2^m} \,\middle|\, A = \mathrm{diag}(a_1, \ldots, a_{2^\ell}),\, a_j \in \mathbf{R}, \sum_{j=1}^{2^\ell} a_j \mu_j = 0 \right\}, \tag{6.9}$$

where $\ell \leqslant m$. Since the metric on $\dot{M}$ is invariant and under the G action, a metric on $G\backslash\dot{M}$ is defined, like (4.58), through

$$\mathrm{d}\sigma^2(\pi_* X, \pi_* Y) = \langle X, Y\rangle_C, \qquad X, Y \in H_C. \tag{6.10}$$

Since $H_C \cong H_{C_0}$ if C and C_0 are sitting in the same G orbit, the metric on $G\backslash\dot{M}$ is put in the form

$$\mathrm{d}\sigma^2 = \mathrm{tr}(\mathrm{d}C_0\, \mathrm{d}C_0^*) = \sum_{j=1}^{2^\ell} \mathrm{d}\mu_j^2, \qquad \frac{1}{2}\,\mathrm{tr}(C\mathrm{d}C^* + \mathrm{d}CC^*) = \sum_{j=1}^{2^\ell} \mu_j\, \mathrm{d}\mu_j = 0, \tag{6.11}$$

which is the standard metric on the sphere $S^{2^{\ell+1}-1}$ restricted to P.

We take up the measure function $F(C)$ defined in (2.8). As is easily verified, the range of F is $0 \leqslant F(C) \leqslant ((2^\ell - 1)/2^\ell)^{2^\ell}$. In particular, the separable and the maximally entangled states are assigned to the minimum and the maximum of F, respectively.

Since F is invariant under the $G = U(2^\ell) \times U(2^m)$ action, it projects to a function $\widetilde{F}$ on $\overline{P}$ through $\widetilde{F}(\boldsymbol{\mu}(C)) = F(C)$. The set of separable states is mapped to $\boldsymbol{e}_1 = (1, \ldots, 0)^T \in \overline{P}$ by π, which is located in the boundary of P. In what follows, we measure the distance between $\boldsymbol{e}_1$ and the level set $\widetilde{F}^{-1}(k)$ with $0 \leqslant k \leqslant ((2^\ell - 1)/2^\ell))^{2^\ell}$ in a similar manner to that for four-qubits. Since the metric on $\overline{P} \subset S^{2^{\ell+1}-1}$ is the standard one, the distance between $\boldsymbol{e}_1$ and $\boldsymbol{x} \in \overline{P}$ is given by $\arccos(\boldsymbol{e}_1 \cdot \boldsymbol{x}) = \arccos x_1$, so that the distance in question is determined by minimizing $\arccos x_1$ on $\widetilde{F}^{-1}(k) \subset \overline{P}$. In other words, our problem is reduced to solving the problem

$$\text{maximize } x_1 \text{ subject to } \sum_{j=1}^{2^\ell} x_j^2 = 1, \qquad \widetilde{F}(\boldsymbol{x}) = k, \tag{6.12}$$

where we do not take into account the restriction $x_1 \geqslant \cdots \geqslant x_{2^\ell} \geqslant 0$ for the time being. Introducing Lagrange's multipliers λ and μ, we consider the function

$$x_1 + \lambda \left(\sum_{j=1}^{2^\ell} x_j^2 - 1 \right) + \mu\big(\big(1 - x_1^2\big) \cdots \big(1 - x_{2^\ell}^2\big) - k\big). \tag{6.13}$$

By differentiation, we can find that necessary conditions for the above function to attain an extremum are $x_2 = \cdots = x_{2^\ell}$, conditions satisfied on the boundary of P. When the extremum occurs, one has $x_1^2 + \cdots + x_{2^\ell}^2 = x_1^2 + (2^\ell - 1)x_2^2 = 1$, so that

$$k = \big(1 - x_1^2\big)\big(1 - x_2^2\big)^{2^\ell - 1} = (2^\ell - 1)x_2^2\big(1 - x_2^2\big)^{2^\ell - 1}, \tag{6.14}$$

which determines x_2 as a function of k. There are two solutions, $x_2^+(k)$ and $x_2^-(k)$, to the above equation in general, where $x_2^+(k) \geqslant x_2^-(k)$. For these values, we have $x_1^\pm(k) = \sqrt{1-(2^\ell-1)\left(x_2^\pm(k)\right)^2}$. This implies that $x_2^-(k)$ yields the maximum, $x_1^-(k)$, of x_1, so that the distance is given by

$$\arccos\sqrt{1-(2^\ell-1)(x_2^-(k))^2}, \tag{6.15}$$

where $x_2^-(k)$ denotes the smaller one of solutions to (6.14). In particular, if $k = ((2^\ell-1)/2^\ell))^{2^\ell}$, then $x_2^\pm(k) = 2^{-\ell/2}$, and thereby equation (6.15) becomes $\arccos 2^{-\ell/2}$, which provides the distance between the set of separable states and the set of maximally entangled states.

6.3. The gradient vector of the measure function

To investigate the monotone of the measure F of entanglement, we wish to study the gradient, ∇F, of F. As is easily verified, with the constraint $\mathrm{tr}(CC^*) = 1$ out of account, the differential of F is given, for $C \in \dot{M}$, by

$$\mathrm{d}F = -F(C)\,\mathrm{tr}((I-CC^*)^{-1}(C\mathrm{d}C^* + \mathrm{d}CC^*)). \tag{6.16}$$

On account of the constraint $\mathrm{tr}(CC^*) = 1$, the tangential component of $\mathrm{d}F$ to $\dot{M}$ is put in the form $\mathrm{d}F + \kappa\,\mathrm{tr}(C\mathrm{d}C^* + \mathrm{d}CC^*)$, where $\kappa \in \mathbf{R}$ is a Lagrange multiplier. The gradient vector ∇F should then take the form $\nabla F = -2F(C)(I-CC^*)^{-1}C + 2\kappa C$. κ is determined by the condition that ∇F be tangent to $\dot{M}$, $\mathrm{tr}(C^*\nabla F + (\nabla F)^*C) = 0$, which results in $\kappa = F(C)(\mathrm{tr}(I-CC^*)^{-1} - 2^\ell)$, so that one obtains

$$\nabla F = 2F(C)(-(I-CC^*)^{-1}C + (\mathrm{tr}((I-CC^*)^{-1}) - 2^\ell)C), \qquad C \in \dot{M}. \tag{6.17}$$

We can easily verify that ∇F is horizontal; $C(\nabla F)^* - (\nabla F)C^* = C^*(\nabla F) - (\nabla F)^*C = 0$.

To describe ∇F more explicitly, we have to evaluate $\mathrm{tr}(I-CC^*)^{-1}$. We here use the singular decomposition $C = UC_0V^*$ with $C_0 = \mathrm{diag}(\mu_1,\ldots,\mu_{2^\ell})$ and $(U,V) \in U(2^\ell)\times U(2^m)$. Since $U^{-1}(I-CC^*)U = I - C_0C_0^*$, one has $U^{-1}(I-CC^*)^{-1}U = (I-C_0C_0^*)^{-1}$, so that

$$\mathrm{tr}((I-CC^*)^{-1}) = \mathrm{tr}((I-C_0C_0^*)^{-1}) = \sum_{j=1}^{2^\ell}\frac{1}{1-\mu_j^2}. \tag{6.18}$$

We here consider the function

$$F_\lambda(C) = \det(\lambda I - CC^*) = \left(\lambda-\mu_1^2\right)\cdots\left(\lambda-\mu_{2^\ell}^2\right), \tag{6.19}$$

where λ is a parameter. Taking the logarithm of this function and differentiating it with respect to λ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\log F_\lambda(C) = \sum_j \frac{1}{\lambda-\mu_j^2}. \tag{6.20}$$

It follows from (6.18) and (6.20) that

$$\mathrm{tr}((I-CC^*)^{-1}) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\log F_\lambda(C)\bigg|_{\lambda=1} = \frac{\frac{\mathrm{d}}{\mathrm{d}\lambda}F_\lambda(C)}{F_\lambda(C)}\bigg|_{\lambda=1}. \tag{6.21}$$

In particular, ∇F is evaluated at C_0 as

$$\nabla F(C_0) = -2F(C_0)\mathrm{diag}\left(\frac{\mu_1}{1-\mu_1^2}, \ldots, \frac{\mu_{2^\ell}}{1-\mu_{2^\ell}^2}\right) + \left(2\,\frac{\mathrm{d}}{\mathrm{d}\lambda}F_\lambda(C_0)\bigg|_{\lambda=1} - 2^{\ell+1}F(C_0)\right)\mathrm{diag}(\mu_1, \ldots, \mu_{2^\ell}). \tag{6.22}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}F_\lambda(C_0)\bigg|_{\lambda=1} = \sum_{j=1}^{2^\ell}\frac{F(C_0)}{1-\mu_j^2}, \tag{6.23}$$

equation (6.22) turns out to be expressed as

$$\nabla F(C_0) = 2F(C_0)\mathrm{diag}\left(\left(\sum_{j\neq k}\frac{1}{1-\mu_j^2} - 2^\ell\right)\mu_k\right). \tag{6.24}$$

Since $\nabla F(C)$ is horizontal, it projects to a vector field on the factor space $G\backslash\dot{M}$. From (6.24), we obtain

$$\pi_*\nabla F(C_0) = 2\widetilde{F}(\boldsymbol{\mu})\left(\left(\sum_{j\neq k}\frac{1}{1-\mu_j^2} - 2^\ell\right)\mu_k\right)^T, \qquad k = 1, \ldots, 2^\ell, \tag{6.25}$$

where μ_k are viewed as the Cartesian coordinates of $\mathbf{R}^{2^\ell}$. It is easy to verify that the right-hand side of the above equation is tangent to P given in (6.7). We point out further that the right-hand side of (6.25) is shown to be the gradient vector of the function $\widetilde{F}(\boldsymbol{\mu})$ on P with respect to the canonical metric on $S^{2^{\ell+1}-1}$.

By solving the equation

$$\sum_{j\neq k}\frac{1}{1-\mu_j^2} - 2^\ell = 0, \qquad k = 1, \ldots, 2^\ell, \tag{6.26}$$

with the constraint $\sum \mu_j^2 = 1$, we find that the critical point of $\pi_*\nabla F(C_0)$ is given by $\mu_j = 2^{-\ell/2}, j = 1, \ldots, 2^\ell$. This shows that $\pi_*\nabla F(C_0)$ has no critical points in P, but has a critical point $(2^{-\ell/2}, \ldots, 2^{-\ell/2})^T$ in the boundary of P. It then turns out that F is monotone on P in the sense that it has no critical points in P.

We now consider the flow defined by

$$\frac{\mathrm{d}\mu_k}{\mathrm{d}t} = 2\widetilde{F}(\boldsymbol{\mu})\left(\sum_{j\neq k}\frac{1}{1-\mu_j^2} - 2^\ell\right)\mu_k. \tag{6.27}$$

The flow on P is approaching to $(2^{-\ell/2}, \ldots, 2^{-\ell/2})^T$, one of boundary points of P, to which maximally entangled states are mapped by π. Further, equation (6.27) is extended to the boundary of P, and the boundary is invariant under the flow. For example, if $\mu_1 = \mu_2 > \cdots > \mu_{2^\ell} > 0$, the condition $\mu_1 = \mu_2$ is shown to be invariant under the flow, and the flow is approaching to the boundary point $(2^{-\ell/2}, \ldots, 2^{-\ell/2})^T$. If $\mu_1 > \cdots > \mu_{2^\ell-1} > \mu_{2^\ell} = 0$, the flow is approaching to a boundary point $(2^{-(\ell-1)/2}, \ldots, 2^{-(\ell-1)/2}, 0)^T$.

6.4. A summary of section 6

Theorem 4. *With respect to the bipartite partition $(\mathbf{C}^2)^{\otimes n} \cong \mathbf{C}^{2^\ell} \otimes \mathbf{C}^{2^m} \cong \mathbf{C}^{2^\ell\times 2^m}$ of an n-qubit system with $n = \ell + m$, $\ell \leqslant m$, the sets of separable states and maximally entangled states* *form submanifolds given by (6.5) and (6.6), respectively. The function $F(C) = \det(I - CC^*)$ defined on the state space serves as a measure of entanglement, which attains the minimal and maximal values on the sets of the separable and maximally entangled states, respectively. The measure function F is monotone in the sense that F has no critical points in the principal stratum $\dot{M}$. The orbit space $G\backslash\dot{M}$ for the principal stratum is diffeomorphic with an open submanifold P of $S^{2^{\ell+1}-1}$, as is given in (6.7) and (6.8), and the Riemannian metric defined on P through the submersion $\pi : \dot{M} \to P$ is isometric with the canonical one on $S^{2^{\ell+1}-1}$ restricted to P. F projects to a function $\tilde{F}$ on P, which is continuously extended to the boundary of P. With respect to the metric on P, the distance between the set $\tilde{F}^{-1}(0)$ and the set $\tilde{F}^{-1}(k)$ with $0 \leqslant k \leqslant \left((2^\ell - 1)/2^\ell\right)^{2^\ell}$ is given by (6.15), which then implies that the state C with $F(C) = k$ is distant from the separable states by the amount given in (6.15).*

As long as the sets of separable states and maximally entangled states and the distance are concerned, this theorem covers the results in the preceding sections. In particular, if $\ell = 1$, the amount given in (6.15) is equal to $\frac{1}{2}\arcsin r$ with $r = 2\sqrt{\det(CC^*)} = 2\sqrt{k}$.

7. Concluding remarks

We make remarks on invariants for $U(2^\ell) \times U(2^m)$ action. For $A = (a_{ij}) \in \mathbf{C}^{2^\ell \times 2^\ell}$, the characteristic polynomial $\det(\lambda I - A)$ is expanded into

$$\det(\lambda I - A) = \sum_{k=0}^{2^\ell} (-1)^k \varphi_k(A)\lambda^k, \tag{7.1}$$

where $\varphi_0(A) = \det A$ and $\varphi_{2^\ell-1}(A) = \operatorname{tr} A$, in particular. φ_k are known as invariants for the $GL(2^\ell, \mathbf{C})$ action, $A \mapsto gAg^{-1}, g \in GL(2^\ell, \mathbf{C})$. We take $A = CC^*$ in (7.1). Since $\det(\lambda I - CC^*)$ is invariant under the $U(2^\ell) \times U(2^m)$ action, $C \mapsto gCh^T, (g, h) \in U(2^\ell) \times U(2^m)$, the coefficients, $\varphi_k(CC^*)$, of λ^k are invariant as well. Thus we have 2^ℓ non-trivial invariants, $\varphi_k(CC^*), k = 0, \ldots, 2^\ell - 1$, with respect to the $U(2^\ell) \times U(2^m)$ action. Two of $\varphi_k(CC^*)$ are $\varphi_0(CC^*) = \det(CC^*)$ and $\varphi_{2^\ell-1}(CC^*) = \operatorname{tr}(CC^*)$. The function $F(C)$ we have worked with in this paper is expressed, in terms of these invariants, as

$$F(C) = \det(I - CC^*) = \det(CC^*) - \varphi_1(CC^*) + \cdots + \varphi_{2^\ell-2}(CC^*), \tag{7.2}$$

where we have already taken into account the constraint $\varphi_{2^\ell-1}(CC^*) = \operatorname{tr}(CC^*) = 1$.

We make further comments on W-states and GHZ-states. For a three-qubit system, the GHZ-state and W-state are given by and belong to

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \in M_2, \tag{7.3}$$

$$\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \in M_1, \tag{7.4}$$

respectively. This means that the GHZ-state is maximally entangled with respect to $(\mathbf{C}^2)^{\otimes 3} \cong \mathbf{C}^2 \otimes \mathbf{C}^4$, but the W-state is not. For a four-qubit system, we have two bipartite partitions. With respect to $(\mathbf{C}^2)^{\otimes 4} \cong \mathbf{C}^2 \otimes \mathbf{C}^8$, the GHZ-state and W-state are given by and belong to

$$\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \in M_2, \tag{7.5}$$

$$\frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \in M_1, \tag{7.6}$$

respectively, where M_2 and M_1 are the sets of states with $\mu_1 = \mu_2 = 1/\sqrt{2}$ and with $\mu_1 > \mu_2 > 0$, respectively. Equations (7.5) and (7.6) mean that the GHZ-state is maximally entangled, but the W-state is not. In contrast with this, with respect to $(\mathbf{C}^2)^{\otimes 4} \cong \mathbf{C}^4 \otimes \mathbf{C}^4$, one has

$$\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \in M_7, \tag{7.7}$$

$$\frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \in M_7, \tag{7.8}$$

where M_7 denotes the set of C with singular values such that $\mu_1 = \mu_2 > \mu_3 = \mu_4 = 0$. This means that neither the GHZ-state nor the W-state is maximally entangled with respect to $(\mathbf{C}^2)^{\otimes 4} \cong \mathbf{C}^4 \otimes \mathbf{C}^4$.

In conclusion, we remark on bipartite and multipartite partitions. Though we have fixed ℓ and m in partitioning the n-qubit, there are many bipartite partitions of the n-qubit, if ℓ and m are not fixed. If ℓ and m are varied under the condition $\ell + m = n$, we have a variety of sets of separable states and maximally entangled states, according to the variation of bipartite partitions. We could have started with a multi-partite partition $\mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$ with local transformation group $U(2) \times \cdots \times U(2)$ and other types of partitions. In this line of thought, we would like to refer to [15–23]. In particular, the space of the three-qubit entanglement types, $S^{15}/(U(2) \times U(2) \times U(2))$, is identified in [21].

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